MATHEMATICS 31

Distance Learning

$(2x^{-3}$

UNIT 1: INTRODUCTION TO DIFFERENTIAL CALCULUS

Alberta EDUCATION

Welcome

Distance Learning

You have chosen an alternate form of learning that allows you to work at your own pace. You will be responsible for your own schedule, for disciplining yourself to study the units thoroughly, and for completing your units regularly. We wish you much success and enjoyment in your studies.

Mathematics 31 Student Module Unit 1 Introduction to Differential Calculus Alberta Distance Learning Centre ISBN No. 0-7741-0241-1

General Information

This information explains the basic layout of each booklet.

- **What You Already Know** and **Review** are to help you look back at what you have previously studied. The questions are to jog your memory and to prepare you for the learning that is going to happen in this unit.

- As you begin each **Topic**, spend a little time looking over the components. Doing this will give you a preview of what will be covered in the topic and will set your mind in the direction of learning.

- **Exploring the Topic** includes the objectives, concept development, and activities for each objective. Use your own papers to arrive at the answers in the activities.

- **Extra Help** reviews the topic. If you had any difficulty with **Exploring the Topic**, you may find this part helpful.

- **Extensions** gives you the opportunity to take the topic one step further.

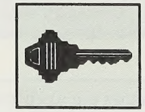
- To summarize what you have learned, and to find instructions on doing the unit assignment, turn to the **Unit Summary** at the end of the unit.

- The **Appendices** include the solutions to **Activities (Appendix A)** and any other charts, tables, etc. which may be referred to in the topics **(Appendix B, etc.)**.

Visual Cues

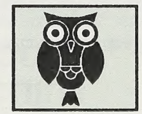
Visual cues are pictures that are used to identify important areas of the material. They are found throughout the booklet.

An explanation of what they mean is written beside each visual cue.

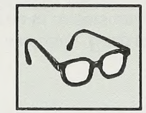


Key Idea
- flagging important ideas

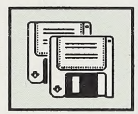
Audiotape
- learning by listening to an audiotape



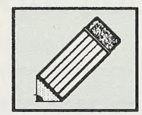
What You Already Know
- reviewing what you already know



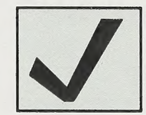
Another View
- exploring different perspectives



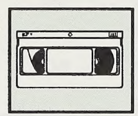
Computer Software
- learning by using computer software



Review
- studying previous concepts



Solutions
- correcting the activities

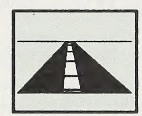


Videotape
- learning by viewing a videotape

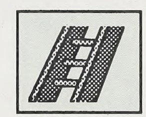
Introduction
- introducing the unit

Extra Help
- providing additional study

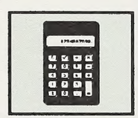
Print Pathway
- choosing a print alternative



What Lies Ahead
- previewing the unit

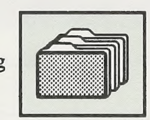


Extensions
- going on with the topic



Calculator
- using your calculator

Exploring the Topic
- actively learning new concepts



What You Have Learned
- summarizing what you have learned

Mathematics 31

Course Overview

Mathematics 31 contains 9 units. Beside each unit is a percentage that indicates what the unit is worth in relation to the rest of the course. The units and their percentages are listed below. You will be studying the unit that is shaded.

Unit 1 Introduction to Differential Calculus	10%
Unit 2 Differentiation of Algebraic Expressions and Graphing	10%
Unit 3 Practical Application of Derivatives	20%
Unit 4 Integration	10%
Unit 5 Geometric Vectors and Their Application	10%
Unit 6 Algebraic Vectors and Their Application	10%
Unit 7 Inner Product	10%
Unit 8 Systems of Linear Equations	10%
Unit 9 Matrices and Linear Transformations	10%
	100%

Unit Assessment

After completing the unit you will be given a mark based totally on a unit assignment. This assignment will be found in the Assignment Booklet.

Unit Assignment - 100%

If you are working on a CML terminal, your teacher will determine what this assessment will be. It may be

Unit Assignment - 50%
Supervised Unit Test - 50%

Introduction to Differential Calculus

This unit covers topics dealing with differential calculus. Each topic contains explanations, examples, and activities to assist you in understanding differential calculus. If you find you are having difficulty with the explanations and the way the material is presented, there is a section called **Extra Help**. If you would like to extend your knowledge of the topic, there is a section called **Extensions**.

You can evaluate your understanding of each topic by working through the activities. Answers are found in the solutions in the **Appendix**. In several cases there is more than one way to do the question.

Unit 1 Introduction to Differential Calculus

Contents at a Glance

Introduction to Differential Calculus

The Latin word for **pebble** is **calculus**. Pebbles were one of the earliest aids to counting.

What is calculus?

Calculus is a branch of mathematics. In the seventeenth century Sir Isaac Newton and Gottfried Leibniz, independent of each other, created calculus.

Calculus has two branches: differential calculus and integral calculus.

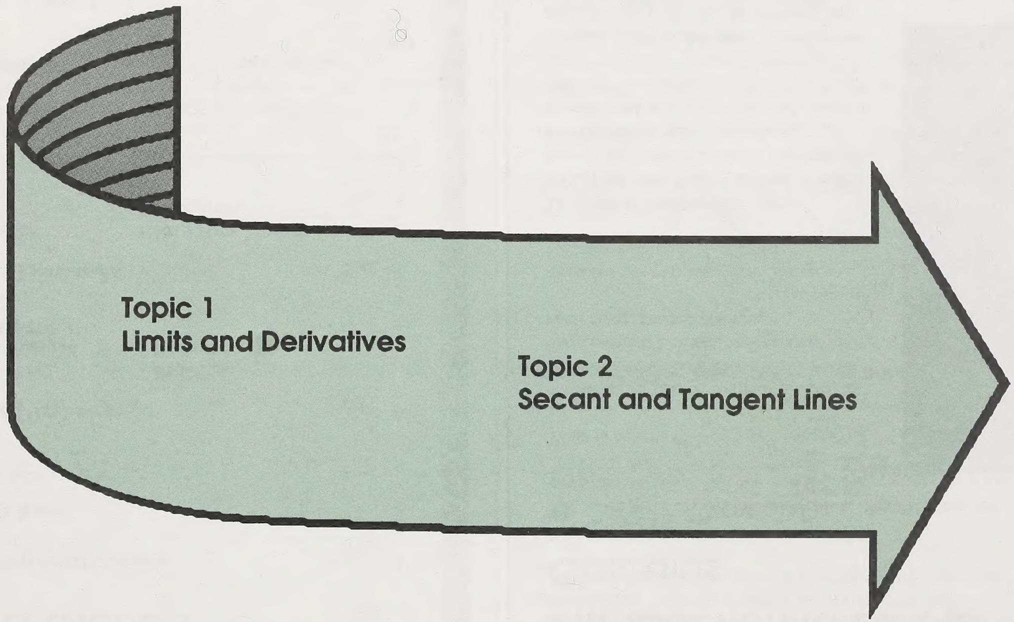
Differential calculus is primarily concerned with finding the rate at which a known but varying quantity changes. It can be used in studying velocity, acceleration, and maximum and minimum values of functions.

Integral calculus, however, is concerned with finding a quantity by knowing the rate at which it is changing. It deals with the calculation of lengths of curves, volumes of solids, and areas of surfaces.

In this unit only differential calculus will be discussed.

Since **limit** is the basic concept in the development of calculus, it will be the first idea studied in this unit.

Unit 1 Introduction to Differential Calculus



What You Already Know

Recall the following.

You should refresh your memory on the concepts listed as follows:

- the laws of exponents (integral and rational)
- operations with polynomials
- methods for factoring polynomials
- operations with rational expressions
- slope, its meaning, and formula
- functional notation
- graphing polynomial functions
- equations of oblique lines and lines parallel to the axis
- conditions for parallel and perpendicular lines

Remember, this is intended to be a refresher.

Review

Try these exercises.

1. Simplify each of the following.

   a. $\dfrac{(2x^2y)(3xy^4)}{(2xy^2)^3}$

   b. $\dfrac{3x^{-1}y^{-2}}{-6xy^{-3}}$

2. Rewrite the following using exponents.

   a. $\sqrt[4]{x^3}$

   b. $\sqrt{x}$

   c. $\dfrac{2}{\sqrt[3]{x}}$

3. Simplify the following.

   a. $(x-3)^2+4x(x-2)-(x-6)^2$

   b. $\left[(x+h)^2-(x+h)\right]-\left(x^2-x\right)$

4. Factor the following.

a. $4x^2 - 4x + 1$ b. $4x^3h^2 - 2h$

c. $4x^2 - 9$ d. $x^{-2} + x^{-5}$

e. $(x+1)^{-3} + (x+1)^{-2}$ f. $(x+2)^{-\frac{1}{2}} + (x+2)^{\frac{1}{2}}$

5. Simplify the following.

a. $\dfrac{x^2 - 6x}{x^2 - 36}$

b. $\dfrac{4x(x-3) - 6(x-3)}{(x-3)^2}$

c. $\dfrac{(x+2)^{-2}(x-6) + (x+2)^{-3}(3x)}{(x+2)^4}$

d. $\dfrac{3x}{x+1} + \dfrac{x+2}{x+3}$

6. Sketch lines through the origin. The lines have the following slopes:

a. 3 b. $-\frac{1}{4}$

c. 0

7. For each set of given points, determine the slope of the line that passes through the points, given $m = \dfrac{y_2 - y_1}{x_2 - x_1}$.

a. $A(2,-3)\ B(3,0)$

b. $A(x, x^2)\ B(x+h, (x+h)^2)$

8. Sketch the following polynomial functions:

a. $P(x) = x^2 - 2x - 8$

b. $P(x) = x^3 + 2x^2 - x - 2$

9. Evaluate $P(x) = x^2 - x + 7$ at the following:

a. $x = 2$ b. $x = -3$

c. $x = x + h$

10. Find the equation of the line which passes through the points (2, 3) and $(5, -8)$.

11. A line passes through the point $(3, -1)$ and has a slope of $\frac{2}{5}$. Find the equation of the line.

12. Find the equation of the line with slope $\frac{1}{3}$ and y-intercept 5.

13. A line passes through the point $P(3,-2)$.

    a. Find the equation of the line when it is parallel to the x-axis.

    b. Find the equation of the line when it is parallel to the y-axis.

14. The equation of a straight line is $3x-y-8=0$.

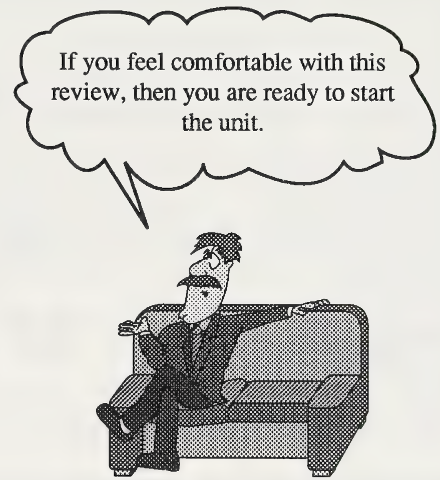
    a. Find the slope of any line perpendicular to $3x-y-8=0$.

    b. Find the slope of any line parallel to $3x-y-8=0$.

15. Find the equation of the line parallel to $2x+y=5$, passing through the point $P(1,-2)$.

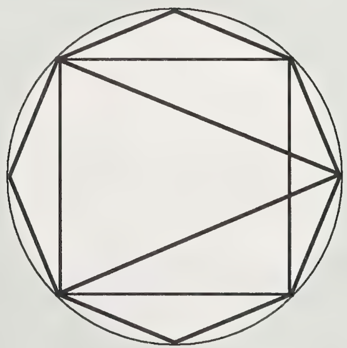
16. Find the equation of the line perpendicular to $3x-2y=2$, passing through the point $P(2,-1)$.

Now go to the **Review** solutions in the **Appendix**.



Topic 1 Limits and Derivatives

Introduction



If you were to calculate the areas of the inscribed polygons in the preceding figure, as the number of sides increased, the area would also increase. Is there any **limit** to the area of the polygon as the number of sides increase? Could the area of the polygon ever exceed that of the circle?

The area of the circle is said to be the limit of the area of the inscribed polygon. As you will see in the following discussion, the concept of limit plays an important role in the development of calculus.

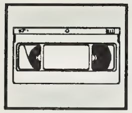
What Lies Ahead

Throughout the topic you will learn to

1. determine the limit of an infinite sequence
2. determine the limit of an algebraic function
3. use the limit theorems

Now that you know what to expect, turn the page to begin your study of limits and derivatives.

Exploring Topic 1



If you have access to a videocassette recorder (VCR), you may wish to view the video titled **Introduction to Calculus and Vectors**. This video is program 1 of the series titled *Catch 31*.[1] This program will give you an overview of some of the concepts covered in this topic.

Activity 1

Determine the limit of an infinite sequence.

The concept of a **limit** is basic to the understanding of calculus. A brief review of sequences and the limits of infinite sequences should help you understand the concept of a limit. The nature and properties of sequences are covered in Mathematics 30 and will not be developed here. You will need to know the following key concepts.

- A sequence is a function having the set of natural numbers as its domain. $(N = \{1, 2, 3, \ldots\})$.

  For example, $f(n) = 3 + n, \quad n \in N$

  $$\therefore f(1) = 3 + 1 = 4$$
  $$f(2) = 3 + 2 = 5$$
  $$f(3) = 3 + 3 = 6$$
  $$\vdots$$

  Therefore, if $f(n) = 3 + n,\ n \in N$ gives the sequence $4, 5, 6, 7, \ldots$.

- The range of a sequence, often referred to as the sequence itself, is any list of elements taken in order.

- If the domain of the sequence is the infinite ordered set N, then the sequence is an **infinite sequence**. (See the example mentioned previously.)

- A **finite sequence** is also possible if it has a limited number of terms. For example, $f(n) = 5n + 2$, where $1 \le n \le 7$, gives the sequence 7, 12, 17, 22, 27, 32, and 37.

$$f(1) = 5 \times 1 + 2 = 7$$
$$f(2) = 5 \times 2 + 2 = 12$$
$$f(3) = 5 \times 3 + 2 = 17$$
$$f(4) = 5 \times 4 + 2 = 22$$
$$f(5) = 5 \times 5 + 2 = 27$$
$$f(6) = 5 \times 6 + 2 = 32$$
$$f(7) = 5 \times 7 + 2 = 37$$

[1] *Catch 31* is a title of ACCESS Network.

- The elements of the domain are mapped onto the elements of the range in one-to-one correspondence.

- The **n^{th} term** of a sequence is denoted by the symbol $f(n)$. For example, in the sequence 7, 12, 17, 22, 27, 32, and 37, note that $f(1)=7, f(2)=12, f(3)=17, f(4)=22, f(5)=27, f(6)=32,$ and $f(7)=37$.

Any finite sequence has a smallest and a largest term. This property does not necessarily hold for an infinite sequence. Consider the sequence $f(n)=\frac{1}{n}$, where $n \in N$.

The sequence is as follows:

$$f(1)=1$$
$$f(2)=\frac{1}{2}=0.5$$
$$f(3)=\frac{1}{3}=0.333\ldots$$
$$f(4)=\frac{1}{4}=0.25$$
$$f(5)=\frac{1}{5}=0.2$$
$$\vdots$$

As n gets greater, $f(n)$ gets smaller. As n increases without bound, $f(n)$ approaches the value of 0. Note that $f(n)$ will never equal nor be less than zero. In this case zero is the limit of this sequence.

Algebraically this can be written as $\lim\limits_{n\to\infty}\frac{1}{n}=0$, where ∞ is the symbol used to denote infinity. You should note that ∞ is not a number, because infinity is undefined.

An infinite sequence is said to be a convergent sequence if it has a limit.

It is important to remember that as $n \to \infty$, the **limit** is **zero** for any rational expression with a constant numerator and a denominator which is a polynomial in n or a power with n in the exponent.

For example, $\lim\limits_{n\to\infty}\frac{1}{n-5}=0, \quad \lim\limits_{n\to\infty}\left[\frac{3}{5^n}\right]=0$

The following examples show you how to apply this property to determine the limit of an infinite sequence.

Example 1

Determine the limit of the sequence $f(n)=\frac{2n-3}{n}$ as $n \to \infty$.

Solution:

$$\begin{aligned}\lim_{n\to\infty}\frac{2n-3}{n} &= \lim_{n\to\infty}\left[\frac{2n}{n}-\frac{3}{n}\right] && \text{(Separate into rational fractions.)}\\ &= \lim_{n\to\infty}\left[2-\frac{3}{n}\right] && \text{(Simplify.)}\\ &= 2-0 && \left(\text{As } n\to\infty, \tfrac{3}{n}\to 0.\right)\\ &= 2\end{aligned}$$

Try another example.

Example 2

Determine the limit of the sequence

$f(x)=\dfrac{3x^2-5x+8}{2x^2+3x-7}$ as $x \to \infty$.

Solution:

$$\lim_{x\to\infty}\frac{3x^2-5x+8}{2x^2+3x-7}=\lim_{x\to\infty}\frac{\cancel{x^2}\left(3-\frac{5}{x}+\frac{8}{x^2}\right)}{\cancel{x^2}\left(2+\frac{3}{x}-\frac{7}{x^2}\right)} \quad \text{(Factor.)}$$

$$=\lim_{x\to\infty}\frac{3-\frac{5}{x}+\frac{8}{x^2}}{2+\frac{3}{x}-\frac{7}{x^2}}$$

$$=\frac{3}{2}$$

$\left(\text{As } x \to \infty,\ \frac{5}{x},\ \frac{8}{x^2},\ \frac{3}{x}, \text{ and } \frac{7}{x^2} \text{ all approach } 0.\right)$

The next example has a power with n in the exponent.

Example 3

Simplify $\lim_{n\to\infty}\left[5-\left(3\times5^{-n}\right)\right]$.

Solution:

$$\lim_{n\to\infty}\left[5-\left(3\times5^{-n}\right)\right]=\lim_{n\to\infty}\left[5-\frac{3}{5^n}\right]$$

$$=5-0 \quad \left(\text{As } n\to\infty, \frac{3}{5^n}\to 0.\right)$$

$$=5$$

Not all sequences approach a limit. Consider the following sequence:

$f(n)=n+3,\ n\in N$

$f(1)=4$

$f(2)=5$

$f(3)=6$

.

.

.

As $n\to\infty, f(n)$ increases without bound. It does not have a limit. **A sequence is said to be a divergent sequence if it does not have a limit.**

Now it is your turn.

Do at least a and c or b and d in the following questions.

1. Give the first five terms of each of the following sequences.

   a. $f(n)=5n-3,\ n\in N$

   b. $f(n)=5-4n,\ n\in N$

   c. $f(n)=2n^2-3,\ n\in N$

   d. $f(n)=3n^2+2,\ n\in N$

2. Give the first five terms of each of the following sequences.

   a. $f(n)=3+\frac{2}{n},\ n\in N$

   b. $f(n)=5-\frac{1}{n},\ n\in N$

   c. $f(n)=3^n+1,\ n\in N$

   d. $f(n)=1-2^n,\ n\in N$

3. Find the stated term of each of the following sequences.

   a. third term if $f(n)=-n^3,\ n\in N$

   b. eighth term if $f(n)=(-2)^n,\ n\in N$

   c. tenth term if $f(n)=\frac{1}{3^n}-1,\ n\in N$

   d. ninth term if $f(n)=3n^2-8,\ n\in N$

4. Find the limit of each of the following sequences as $n\to\infty$.

   a. $\lim\limits_{n\to\infty}\left[\frac{2n-3}{n}\right]$

   b. $\lim\limits_{n\to\infty}\left[\frac{5+3n}{n}\right]$

   c. $\lim\limits_{n\to\infty}\left[\frac{3n^3-2n+1}{4n^3-2n^2+n-7}\right]$

   d. $\lim\limits_{n\to\infty}\left[\frac{n^3-8n^2+3n+1}{n^3-2n+8}\right]$

5. Find the limit of each of the following sequences as $n \to \infty$.

   a. $\lim\limits_{n\to\infty}\left[2\left(3^{n}-1\right)\right]$

   b. $\lim\limits_{n\to\infty}\left[5-\left(3+2^{n}\right)\right]$

   c. $\lim\limits_{n\to\infty}\left[3^{-n}+5\right]$

   d. $\lim\limits_{n\to\infty}\left[\dfrac{\frac{2}{3}}{2-\left(\frac{1}{2}\right)^{n}}\right]$

For solutions to **Activity 1**, turn to the **Appendix, Topic 1**.

Activity 2

Determine the limit of an algebraic function.

You have discussed the limit of a sequence as $n \to \infty$. You will now extend your knowledge to any rational algebraic function. Hence, you will extend the domain of the variable to the set of real numbers. You will also evaluate limits, not just for the variable increasing without bound, but for the variable approaching any real value.

What is an algebraic function? Algebraic functions consist of polynomial functions, rational algebraic functions, and functions containing roots or fractional exponents. They will be the focus of your study in calculus. A rational algebraic function is of the form $\frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials in x. You will discuss the limits of such functions as x approaches some real number a. If the function exists at $x = a$, then the problem of finding its limit as x approaches a is trivial. You merely evaluate the function at $x = a$. Look at the following example.

Nonalgebraic functions (transcendental functions) include the trigonometric, exponential, and logarithmic functions.

Example 4

Find the limit of the function $y = 2x^2 - 3x + 5$ as x approaches 2.

Solution:

$$\lim_{x \to 2}\left[2x^2 - 3x + 5\right] = 2(2)^2 - 3(2) + 5$$
$$= 7$$

Look at the limit of a rational algebraic function.

Example 5

Find the limiting value of $y = \frac{x^2-9}{x-3}$ as x approaches 3.

Solution:

$f(x) = \dfrac{x^2 - 9}{x - 3}$ is defined for all real values of x except 3.

At $x = 3$, $f(x)$ is undefined because division by zero is undefined.

$$\therefore f(x) = \frac{x^2 - 9}{(x-3)}, \quad x \neq 3$$

$$= \frac{(x+3)(x-3)}{(x-3)} \quad \text{(Factor.)}$$

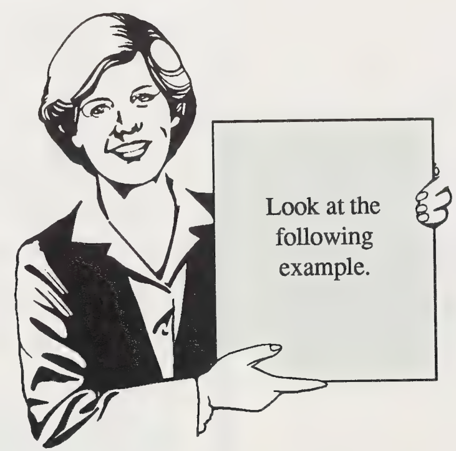
$$= (x+3) \quad \text{(You can reduce the factor } (x-3) \text{ only if } x \neq 3.)$$

Since $\lim_{x \to 3}(x+3) = 6$,

$$\lim_{x \to 3} \frac{(x^2 - 9)}{(x-3)} = \lim_{x \to 3} \frac{(x+3)\cancel{(x-3)}}{\cancel{(x-3)}}$$
$$= \lim_{x \to 3}(x+3)$$
$$= 6$$

(Note that $(x-3)$ can be reduced because x is approaching but not equal to 3.)

If you use the symbol L to denote the limit of $f(x)$ as $x \to k$, you can always find a value of x which is very close to k, such that $f(x)$ differs from L by a very small quantity.



Example 6

For what value of n does $\frac{n^2-9}{n-3}$ differ from its limit by less than 10^{-5} as $n \to 3$?

Solution:

$$L = \lim_{n\to 3}\left[\frac{n^2-9}{n-3}\right]$$
$$= \lim_{n\to 3}(n+3)$$
$$= 6$$

Now $|f(n)-L| < 10^{-5}$, $n \neq 3$

$$\left|\frac{n^2-9}{n-3}-6\right| < 10^{-5}$$
$$|n+3-6| < 10^{-5}$$
$$|n-3| < 10^{-5}$$

$$3-10^{-5} < n < 3+10^{-5}$$

The next example has a limiting value when $n \to \infty$.

Example 7

For what values of n does $2+\frac{1}{n^2}$ differ from its limit by less than 10^{-4} as $n \to \infty$?

Solution:

$$L = \lim_{n\to\infty}\left[2+\frac{1}{n^2}\right]$$
$$= 2$$

$$|f(n)-L| < 10^{-4}$$
$$\left|2+\frac{1}{n^2}-2\right| < 10^{-4}$$
$$\left|\frac{1}{n^2}\right| < 10^{-4}$$
$$\frac{1}{n^2} < 10^{-4}$$
$$\frac{10^4}{n^2} < 1, \text{ since } 10^4\left(\frac{1}{n^2}\right) < 10^{-4}\left(10^4\right) \text{ and } 10^0 = 1$$
$$10^4 < n^2$$
$$n > 10^2$$
$$n > 100$$

Now you need some practice.

Do at least the odd-numbered questions.

1. Find the limit L if $L = \lim\limits_{x \to -3} \left[\dfrac{x^2 - 9}{x + 3} \right]$.

2. Find the limit L if $L = \lim\limits_{x \to 1} \left[\dfrac{x^2 + 7x - 8}{x - 1} \right]$.

3. Find the limit L if $L = \lim\limits_{x \to 3} \left[\dfrac{x^2 + 2x - 15}{x - 3} \right]$.

4. Find the limit L if $L = \lim\limits_{x \to 2} \left[\dfrac{5x - 10}{x - 2} \right]$.

5. Find the limit L if $L = \lim\limits_{x \to 1} \left[\dfrac{6x - 2}{3x - 1} \right]$.

6. Find the limit L if $L = \lim\limits_{x \to 1} \left[\dfrac{x - 1}{x^2 - 1} \right]$.

7. Find the limit L if $L = \lim\limits_{x \to 2} \left[\dfrac{x + 2}{x - 3} \right]$.

8. Find the limit L if $L = \lim\limits_{x \to 5} \left[\dfrac{x^2 - x - 2}{x - 5} \right]$.

9. Find the limit L if $L = \lim\limits_{x \to 9} \left[\dfrac{x^2 - 1}{x - 9} \right]$.

10. Find the limit L if $L = \lim\limits_{x \to 0} \left[\dfrac{3 - \frac{1}{x}}{\frac{2}{x} - 5} \right]$.

11. Find the limit L if $L = \lim\limits_{x \to 3} \left[\dfrac{x^3 - 27}{x - 3} \right]$.

12. For what values of x does $\frac{x^2 - 16}{x - 4}$ differ from its limit by less than 10^{-6} as $x \to 4$?

13. For what values of x does $\frac{x^2 - 25}{x + 5}$ differ from its limit by less than 10^{-8} as $x \to 5$?

For solutions to **Activity 2**, turn to the **Appendix, Topic 1**.

Activity 3

Use the limit theorems.

To make limits more useful, you must be able to conduct routine calculations with them. The following is a list of the limit theorems.

Theorem 1: $\lim\limits_{x \to a}\left[f(x)+g(x)\right]=\lim\limits_{x \to a} f(x)+\lim\limits_{x \to a} g(x)$

Theorem 2: $\lim\limits_{x \to a}\left[f(x)-g(x)\right]=\lim\limits_{x \to a} f(x)-\lim\limits_{x \to a} g(x)$

Theorem 3: $\lim\limits_{x \to a}\left[f(x)\bullet g(x)\right]=\lim\limits_{x \to a} f(x)\bullet\lim\limits_{x \to a} g(x)$

Theorem 4: $\lim\limits_{x \to a} cf(x)=c\lim\limits_{x \to a} f(x)$, where c is a constant

Theorem 5: $\lim\limits_{x \to a}\left[\dfrac{f(x)}{g(x)}\right]=\dfrac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)}$, if $\lim\limits_{x \to a} g(x)\neq 0$

Theorem 6: $\lim\limits_{x \to a} c=c$, where c is a constant

Use these theorems to verify the following questions.

(Note that in Theorem 3 the symbol • represents multiplication.)

Example 8

Verify that $\lim\limits_{x \to 2}\left[(2x+1)+(3x-4)\right]=\lim\limits_{x \to 2}(2x+1)+\lim\limits_{x \to 2}(3x-4)$.

Solution:

LS	RS
$\lim\limits_{x \to 2}\left[(2x+1)+(3x-4)\right]$	$\lim\limits_{x \to 2}(2x+1)+\lim\limits_{x \to 2}(3x-4)$
$\lim\limits_{x \to 2} 5x-3$	$2(2)+1+2(3)-4$
$5(2)-3$	$4+1+6-4$
$10-3$	7
7	7

LS = RS

Example 9

Verify that $\lim\limits_{x \to 1}\left[(x-3)(x+2)\right]=\lim\limits_{x \to 1}(x-3)\bullet\lim\limits_{x \to 1}(x+2)$.

Solution:

LS	RS
$\lim\limits_{x \to 1}\left[(x-3)(x+2)\right]$	$\lim\limits_{x \to 1}(x-3)\bullet\lim\limits_{x \to 1}(x+2)$
$\lim\limits_{x \to 1} x^2-x-6$	$(1-3)(1+2)$
$1-1-6$	$(-2)(3)$
-6	-6

LS = RS

Example 10

Verify that $\lim\limits_{x\to3} 5(x+7) = 5\lim\limits_{x\to3}(x+7)$.

Solution:

LS	RS
$\lim\limits_{x\to3} 5(x+7)$	$5\lim\limits_{x\to3}(x+7)$
$\lim\limits_{x\to3} 5x+35$	$5(3+7)$
$5(3)+35$	5×10
50	50

LS = RS

Example 11

Verify that $\lim\limits_{x\to3}\left[\frac{x-1}{x+2}+\frac{x^2-9}{x-3}\right] = \lim\limits_{x\to3}\left[\frac{x-1}{x+2}\right]+\lim\limits_{x\to3}\left[\frac{x^2-9}{x-3}\right]$.

Solution:

LS	RS
$\lim\limits_{x\to3}\left[\frac{x-1}{x+2}+\frac{x^2-9}{x-3}\right]$	$\lim\limits_{x\to3}\left[\frac{x-1}{x+2}\right]+\lim\limits_{x\to3}\left[\frac{x^2-9}{x-3}\right]$
$\lim\limits_{x\to3}\left[\frac{(x-1)}{(x+2)}+\frac{(x+3)\cancel{(x-3)}}{\cancel{(x-3)}}\right]$	$\lim\limits_{x\to3}\left[\frac{x-1}{x+2}\right]+\lim\limits_{x\to3}\frac{(x+3)\cancel{(x-3)}}{\cancel{(x-3)}}$
$\lim\limits_{x\to3}\left[\frac{x-1}{x+2}+x+3\right]$	$\frac{3-1}{3+2}+(3+3)$
$\frac{3-1}{3+2}+3+3$	$\frac{2}{5}+6$
$6\frac{2}{5}$	$6\frac{2}{5}$

LS = RS

Do at least any two of the following questions.

1. Verify that $\lim\limits_{x\to5}\left[\frac{x^2-25}{x+5}\right]\left[\frac{x^2-25}{x-5}\right]=\lim\limits_{x\to5}\left[\frac{x^2-25}{x+5}\right]\bullet\lim\limits_{x\to5}\left[\frac{x^2-25}{x-5}\right]$.

2. Verify that $\lim\limits_{x\to3}\left[\dfrac{\frac{x^2-9}{x+3}}{\frac{x^2-9}{x-3}}\right]=\dfrac{\lim\limits_{x\to3}\left[\frac{x^2-9}{x+3}\right]}{\lim\limits_{x\to3}\left[\frac{x^2-9}{x-3}\right]}$.

3. Verify that $\lim\limits_{x\to2}(2)\left(\frac{x^2-4}{x-2}\right)=2\lim\limits_{x\to2}\left[\frac{x^2-4}{x-2}\right]$.

4. Verify that $\lim\limits_{x\to1}\left[\frac{x^2-4x+3}{x-1}-\frac{x^2+x-2}{x-1}\right]=\lim\limits_{x\to1}\left[\frac{x^2-4x+3}{x-1}\right]-\lim\limits_{x\to1}\left[\frac{x^2+x-2}{x-1}\right]$.

For solutions to **Activity 3**, turn to the **Appendix, Topic 1**.

If you require help, do the Extra Help section.

If you want more challenging explorations, do the Extensions section.

} You may decide to do both.

Extra Help

To determine the limit of a function, you can substitute a into the limit expression if $x \to a$. If the limit expression is a rational expression, then factor it completely and reduce all common factors which appear in both numerator and denominator before you evaluate the limit expression.

Example 12

Find the limit of $2x-3$ as $x \to 3$.

Solution:

$$\begin{aligned}\lim_{x\to 3}(2x-3) &= 2(3)-3\\ &= 3\end{aligned}$$

Example 13

Find the limit of $\frac{x^2-25}{x+5}$ as $x \to -5$.

Solution:

$$\begin{aligned}\lim_{x\to -5}\frac{x^2-25}{x+5} &= \lim_{x\to -5}\frac{\cancel{(x+5)}(x-5)}{\cancel{(x+5)}}\\ &= \lim_{x\to -5}(x-5)\\ &= -5-5\\ &= -10\end{aligned}$$

If you wish to find a value of x such that $f(x)$ differs from its limit by a very small quantity, look at the example which follows.

Example 14

For what value of x does $(x-3)$ differ from its limit by less than 10^{-5} as $x \to 4$?

Solution:

$$\lim_{x \to 4}(x-3) = 4-3$$
$$= 1$$
$$|f(x)-L| = |(x-3)-1|$$

If $|(x-3)-1| < 10^{-5}$,

then $|x-4| < 10^{-5}$

$4-10^{-5} < x < 4+10^{-5}$.

The limit theorems you need are listed next.

Theorem 1: The limit of the sum of two functions $f(x)$ and $g(x)$ is the limit of $g(x)$ added to the limit of $f(x)$ as $x \to a$.

Theorem 2: The limit of the result of subtracting function $g(x)$ from $f(x)$ is the limit of $g(x)$ subtracted from the limit of $f(x)$ as $x \to a$.

Theorem 3: The limit of the product of two functions $f(x)$ and $g(x)$ is the limit of $g(x)$ multiplied by the limit of $f(x)$ as $x \to a$.

Theorem 4: The limit of the product of a constant c and a function $f(x)$ is the limit of $f(x)$ multiplied by the constant c as $x \to a$.

Theorem 5: The limit of the result of dividing function $f(x)$ by function $g(x)$ is the limit of $f(x)$ divided by the limit of $g(x)$ as $x \to a$.

Theorem 6: The limit of a constant c is c as $x \to a$.

If $|x| < a$, then $-a < x < a$.
Therefore, if $|x-4| < 10^{-5}$,
then $-10^{-5} < x-4 < 10^{-5}$ or
$4-10^{-5} < x < 4+10^{-5}$.

Try the following questions.

1. Determine the limit of each of the following:

   a. $\lim\limits_{n\to\infty}\left[\dfrac{n-3}{n+2}\right]$

   b. $\lim\limits_{n\to\infty}\left[5-\dfrac{1}{n^2}\right]$

   c. $\lim\limits_{x\to 8}\dfrac{\left(x^2-64\right)}{(x-8)}$

   d. $\lim\limits_{x\to 3}\left[\left(\dfrac{x}{3}\right)\left(\dfrac{x^2-9}{x-3}\right)\right]$

2. For what value of x does $(x+5)$ differ from its limit by less than 10^{-2} as $x\to 3$?

For solutions to **Extra Help**, turn to the **Appendix, Topic 1.**

Extensions

Do you remember geometric sequences from Mathematics 30? Compound interest can be represented by a finite geometric series. A geometric sequence is a sequence in which the ratio between two consecutive terms is constant. For example, 2, 4, 8, 16, . . . is a geometric sequence because the ratio between any two consecutive terms is always equal to two.

The sum of all terms in a sequence is called a **series**. A geometric series of n terms can be written in the form

$a+ar+ar^2+ar^3+\ .\ .\ .\ +ar^{n-1}$, where a is the first term and r is the common ratio. The sum of the first n terms is given by

$$S_n=\frac{a\left(1-r^n\right)}{1-r}=\frac{a}{1-r}-\frac{ar^n}{1-r}, \text{ where } r\neq 1.$$

What happens to r^n affects S_n. If $|r|<1$, then as $n\to\infty$, $r^n\to 0$ and thus $\dfrac{ar^n}{1-r}\to 0$. Therefore, $\lim\limits_{n\to\infty}S_n=\dfrac{a}{1-r}$. Use the symbol S to denote this limit.

$$S=\frac{a}{1-r}$$

If $|r|>1$, then $\lim\limits_{n\to\infty}r^n$ does not exist and S does not exist.

If $r=1$, S_n is undefined $(1-r=0)$. The series becomes $a+a+a+a+\ .\ .\ .\ .$ Therefore, $S_n=na$, but S does not exist.

If $r=-1$, the series becomes
$a-a+a-a+a-a+\ .\ .\ .\ .$
If n is odd, $S_n = a$. If n is even, $S_n = 0$.
S does not exist.

Now if you want to find the number of terms such that $0.02+0.002+0.0002+\ .\ .\ .$ differs from its limit L by less than 10^{-8} as the number of terms $n \to \infty$, you have to find $|S_n - L|$ first.

$$S_n = \frac{a(1-r^n)}{1-r},\ a=0.02,\ r=\frac{1}{10}$$
$$= \frac{(0.02)\left(1-\left(\frac{1}{10}\right)^n\right)}{1-\frac{1}{10}}$$
$$= \frac{(0.02)\left(1-\frac{1}{10^n}\right)}{\frac{9}{10}}$$
$$= \frac{0.2}{9}\left(1-\frac{1}{10^n}\right)$$
$$= \frac{0.2}{9}-\frac{0.2}{9\times10^n}$$

$$L = \lim_{n\to\infty} S_n = \frac{a}{1-r}$$
$$= \frac{0.02}{1-\frac{1}{10}}$$
$$= \frac{0.02}{\left(\frac{9}{10}\right)}$$
$$= \frac{0.2}{9}$$

Since $|S_n - L| < 10^{-8}$

$$\left|\cancel{\frac{0.2}{9}} - \frac{0.2}{9\times10^n} - \cancel{\frac{0.2}{9}}\right| < 10^{-8}$$
$$\left|\frac{-0.2}{9\times10^n}\right| < \frac{1}{10^8}$$
$$\left|\frac{-1}{45\times10^n}\right| < \frac{1}{10^8},$$

then $n=7$.

Now try the following question.

Find the number of terms such that the series $1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\ .\ .\ .$ differs from its limit by less than 3^{-9} as the number of terms $n \to \infty$.

For solutions to **Extensions**, turn to the **Appendix, Topic 1**.

Since $45 > 10$ and $\frac{1}{45} < \frac{1}{10}$,

then $\frac{1}{45\times10^7} < \frac{1}{10\times10^7}$

$\frac{1}{45\times10^7} < \frac{1}{10^8}$.

Topic 2 Secant and Tangent Lines

Introduction

A curve may or may not be straight. When it is straight, it can be considered to have a constant slope.

If you draw the graph of a quadratic or cubic function, you will obtain a curved line. You can relate this to the speed of a particle in motion. As long as the speed remains fixed, the path of the particle will be a straight line. However, if the speed of this particle begins to vary, the line will no longer be straight. For the curve, the secant line gives the average velocity of that particle in motion. Actually the secant line is the line of best fit for the motion of the particle for a given period of time. The slope of the line called the tangent is the result when the speed of the particle is taken at any particular instant.

What Lies Ahead

Throughout the topic you will learn to

1. identify secant lines by finding their slopes and writing their equations
2. define and identify tangent lines
3. define and determine the first derivative of a function
4. determine the equations of tangent lines to algebraic functions

Now that you know what to expect, turn the page to begin your study of secant and tangent lines.

Exploring Topic 2

Activity 1

Identify secant lines by finding their slopes and writing their equations.

Up to now you have been dealing with the slope of a straight line. The slope of a straight line is the same all the way along the line, and it is quite simple to determine. You can find the slope of a straight line either from the equation of the line or from the coordinates of any two points on the line.

If the equation of the function cannot be put in the form $Ax+By=C$, the graph is not a straight line, but a curve. The slope is not constant for the entire curve, but changes at each successive value of x. When dealing with curves, you cannot simply pick two points on the curve and obtain the slope of the curve, or any part of the curve, in the same way that you obtain the slope of a straight line. The only simple method of dealing with the slope of a curve involves calculus. The first step is to make sure you can identify points on the curve.

Consider the point $A(1,1)$. Does this point lie on the line $y=x^2-2x+2$? In order to answer this question, substitute for x and y.

$$\begin{aligned} y &= x^2-2x+2 \\ A(1,1),\ (1) &= (1)^2-2(1)+2 \\ 1 &= 1 \end{aligned}$$

Since the left side is the same as the right side, point A lies on the curve $y=x^2-2x+2$.

If the sides were not equal, the point would not be on the curve.

A function of x has a graph which is either a straight line or a curve. If the graph is a curve, you can draw as many lines as you wish through any one point on that curve. One of these lines will be perpendicular to the x-axis. It cannot possibly pass through any other point on the curve because no two ordered pairs of a function have the same first component.

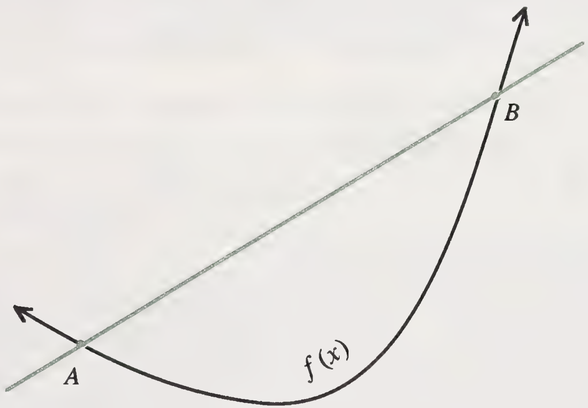
For now, a secant is the line that cuts a curve in two or more distinct places. You will learn a more formal definition later.

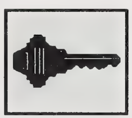
You will need to be able to find the slope of a secant line to a curve. In order to do this, you apply the following formula:

$$m=\frac{y_2-y_1}{x_2-x_1}$$

Remember that the secant cuts the curve in at least two places. The following diagram gives you an idea of what a secant should look like.

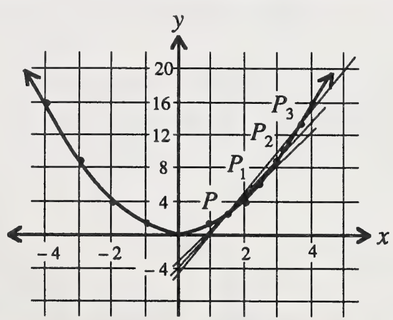
The secant here is AB and the curve is $f(x)$.





A **secant** is any straight line that intersects a curve in two or more distinct places.

If the function of a graph produces a curve, you can draw as many straight lines as you wish through any point on the curve.



The previous graph illustrates the function $y = x^2$ for $-4 \le x \le 4,\ x \in R$.

Note that by keeping a fixed point P on the curve and choosing other points $P_1, P_2, P_3, \ldots$ on the curve, you can calculate the slope of each secant.

The slope of the secant PP_1 is less than that of PP_2, and the slope of PP_3 is greater than that of PP_2, and so on. This is true for all curves. Therefore, the straight line is the only curve function that has a constant slope. In other words, the curvature of the straight line is zero. Try finding the slope of a secant.

Example 1

A line intersects a curve at the points $A(5,1)$ and $B(2,-5)$. What is the slope of the secant AB? Assume that the curve is smooth and continuous.

Solution:

$$m = \frac{y_2 - y_1}{x_2 - x_1} \quad (m \text{ will denote the slope of the line.})$$

$$= \frac{-5-1}{2-5}$$

$$= \frac{-6}{-3}$$

$$= 2$$

The slope of the secant AB is 2.

Example 2

Find the slope of the secant of $y=2x^4$ through each pair of points at which x has the values 1 and 2 respectively.

Solution:

You substitute $x=1$ and $x=2$ in $y=2x^4$ to get the corresponding values of y.

When $x_1=1,\ y_1=2(1)^4$

$$\begin{aligned} &= 2(1) \\ &= 2 \end{aligned}$$

When $x_2=2,\ y_2=2(2)^4$

$$\begin{aligned} &= 2(16) \\ &= 32 \end{aligned}$$

$$m=\frac{y_2-y_1}{x_2-x_1}$$

$$m=\frac{32-2}{2-1}$$

$$=\frac{30}{1}$$

$$=30$$

The slope of the secant is 30.

Example 3

Determine the equation of the secant of the graph $y=5x^2$ between the points for which the abscissae are -1 and 2.

Solution:

First of all, substitute $x=-1$ and $x=2$ in $y=5x^2$ to get the corresponding values of y.

When $x_1=-1,\ y_1=5(-1)^2=5(1)=5$

When $x_2=2,\ y_2=5(2)^2=5(4)=20$

Secondly,

$$\frac{y-y_1}{x-x_1}=\frac{y_2-y_1}{x_2-x_1}$$

$$\frac{y-5}{x-(-1)}=\frac{20-5}{2-(-1)}$$

$$\frac{y-5}{x+1}=\frac{15}{3}$$

$$\frac{y-5}{x+1}=\frac{5}{1}$$

$$5(x+1)=1(y-5)$$

$$5x+5=y-5$$

$$5x-y+10=0$$

Alternate method:

$$Ax+By=C,\ \frac{-A}{B}=m$$

$$\frac{y-5}{x+1}=\frac{5}{1}=m$$

$$\frac{-A}{B}=\frac{A}{-B}=\frac{5}{-(-1)}$$

Thus, $A=5$ and $B=-1$.

$$Ax+By=C$$

$$5x-y=C$$

Substitute $(-1,5)$ for (x,y).

$$5(-1)-5=C$$

$$-10=C$$

$$5x-y=-10$$

The equation of the secant is $5x-y=-10$.

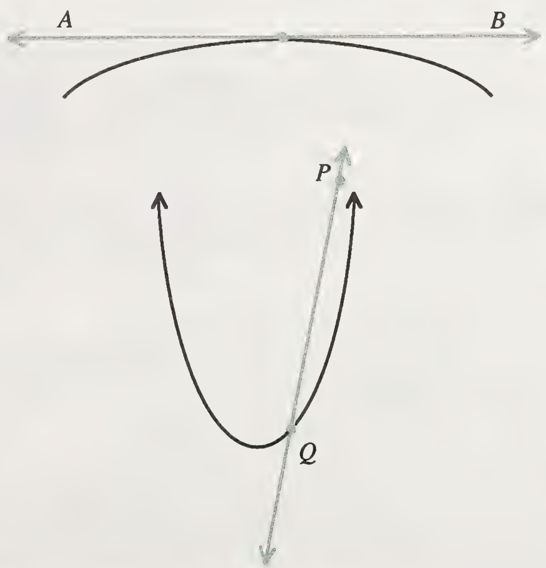
Do any four of the following questions.

1. a. Draw two curves showing the secant line cutting one curve in two places.

   b. Draw two curves showing the secant line cutting one curve in more than two places.

2. a. Why can't you draw a secant line that intersects a curve at one point?

   b. Why are the following lines, AB and PQ, not considered secants?



3. Points $A(1,-2)$ and $B(2,7)$ are on the graph of the curve $y=2x^3-3x^2+4x-5$. Find the slope of the secant AB.

4. Find the slope of the secant of $y=5x^3-3x^2-4x+7$ through each pair of points at which x has the values 1 and -1 respectively.

5. Derive the equations of the secants of the graph $y=x^2-3$ between the points which have ordinates at 1 and 6. Determine the equation for each secant on either side of the y-axis.

6. a. Find the slope of the secants of the curve $y=x^2$ through the point $A(2,4)$ and the points for which the abscissae are 1.9, 1.99, 1.999 . . . .

   b. If the slopes form a sequence, state the likely slope of the line through A.

   c. If $A(2,4)$ is the point (a,b), what is the slope as a multiple of a?

For solutions to **Activity 1**, turn to the **Appendix, Topic 2**.

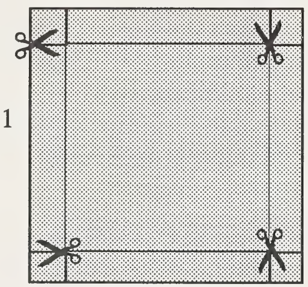
Activity 2

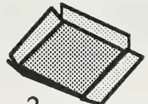
Define and identify tangent lines.

If you cut equal squares from each corner of a square sheet of paper and then fold up the sides and join the edges, you make a rectangular box. If you cut a larger square from each corner of that piece of paper, would the volume of the box you make be larger or smaller than the first box? How can you construct the box of maximum volume?

To gain an understanding of this problem, construct a few boxes. Use square sheets of paper which have sides of 15 cm.

1





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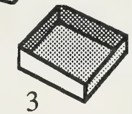

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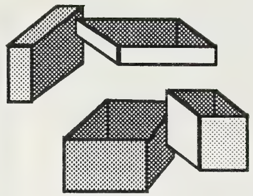


Now investigate the volume of each box. A record of your observations might look like this:

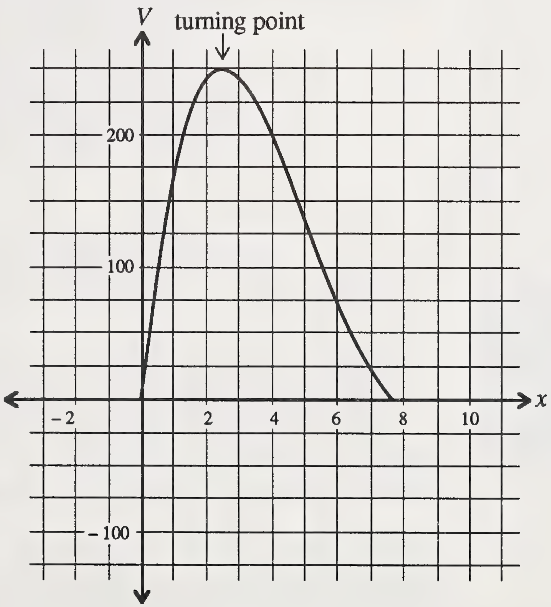
Cut out (cm/side)	Length (cm)	Width (cm)	Height (cm)	Volume (cm^3)
1	13	13	1	169
2	11	11	2	242
3	9	9	3	243
4	7	7	4	196
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
x	$(15-2x)$	$(15-2x)$	x	$(15-2x)(15-2x)(x)$

In this investigation you can see that the volume of each box is given by the polynomial function $V=(15-2x)(15-2x)(x)$, where $0<x\leq 7.5$.

The variable x is the independent variable and the volume (V) is the dependent variable. Simplify the function.

$$V = (15-2x)(15-2x)(x),\ 0 < x \le 7.5$$
$$= \left(225-60x+4x^2\right)x$$
$$= 4x^3-60x^2+225x,\ 0 < x \le 7.5$$

Using techniques from Graphing of Polynomial Functions in Mathematics 30 and the values established in your investigation, you can sketch the graph of the function.

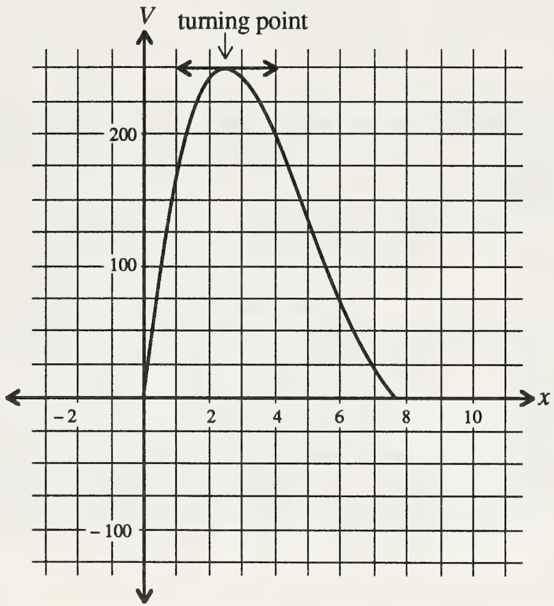


In the function, x can be any value between 0 and 7.5 inclusive. In this investigation the cuts were 1 cm long.

It appears from the graph that the maximum volume occurs at the turning point of the graph, when x is approximately 2.5 cm. If you want a more accurate maximum value, you can investigate more closely the interval around 2.5. Let x be 2.4, 2.5, and 2.6 and calculate the volume (V).

x	V
2.4	249.696
2.5	250.000
2.6	249.704

How can you be sure that 250 is the maximum volume when $x = 2.5$? The method you used to find the maximum is not very accurate and is very time-consuming. Take a closer look at the graph.



Maximum volume occurs at the turning point of the graph. This observation and the further observation that the tangent line at this turning point is horizontal were made by Fermat in the seventeenth century.

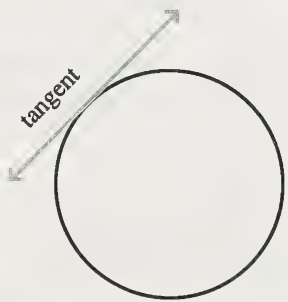
It is clear that the maximum or minimum value of a curve occurs at the turning point of the graph and if you draw a tangent at this turning point, the tangent is always a horizontal line. The slope of a horizontal line is zero.

How can this kind of information help you determine the maximum or minimum value of a curve? The search for the relationship between a function, the slope of a function, and the slope of the tangent to the curve defined by that function led to the development of differential calculus by Newton and Leibniz. You are about to follow their footsteps.

If two points are given, you can determine the slope of a line joining these two points. How do you determine the slope of a tangent to a curve at a given point?

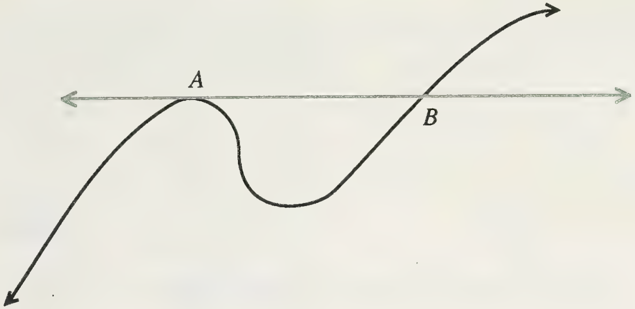
To answer this question, it is necessary to have a closer look at tangents.

Consider a circle. A tangent to the circle is a line that touches the circle at a single point.



A tangent is a straight line touching (not intersecting) a curve at a single point. Sometimes the curve continues in such a way that the tangent may intersect the curve at another point.

However, when you consider tangents to curves other than circles, a new definition is needed. Look at the following diagram. The tangent touches the curve at A and intersects the curve at B.

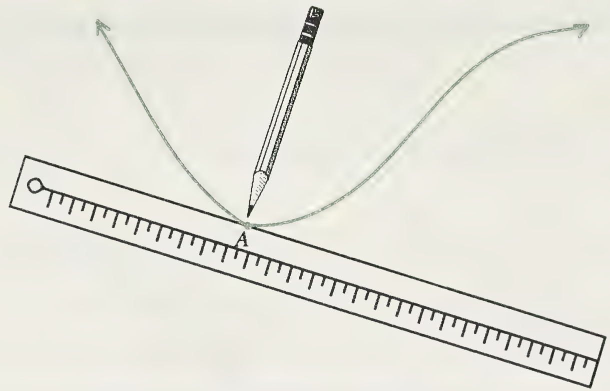


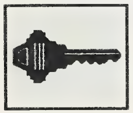
What definition can be used? Look at the next section and follow the ideas. A definition of a tangent is being developed.

Now draw any smooth continuous curve. Choose a point on that curve and label the point A. Now draw a line which is tangent to your curve at point A.

To draw the tangent, you probably placed your ruler on point A and moved it until it **did not** touch the curve at more than one point in the neighbourhood of A.

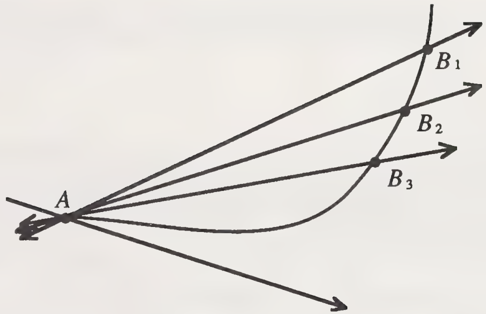
This procedure indicates a much better definition of a tangent and introduces you to one of the most important concepts of calculus, namely, the **limit**.





If a line passes through two distinct points on a curve, it is called a **secant line**. In your drawing the initial positions of the ruler could represent secants. The **tangent** line at any point A on a curve is the **limiting** position of a sequence of secant lines containing A. If you go past that position, the line is once again a secant line.

A tangent to the curve at point A is the limiting position of a sequence of secants containing point A and points $B_1, B_2, B_3, \ldots$ such that B_n is approaching A; that is, the distance AB_n is approaching zero.



Since secants pass through two points on a curve, you can determine the slope of any secant line.

As B_n approaches A, the slope of the secant will approach the value which is the slope of the tangent. Look at the following example.

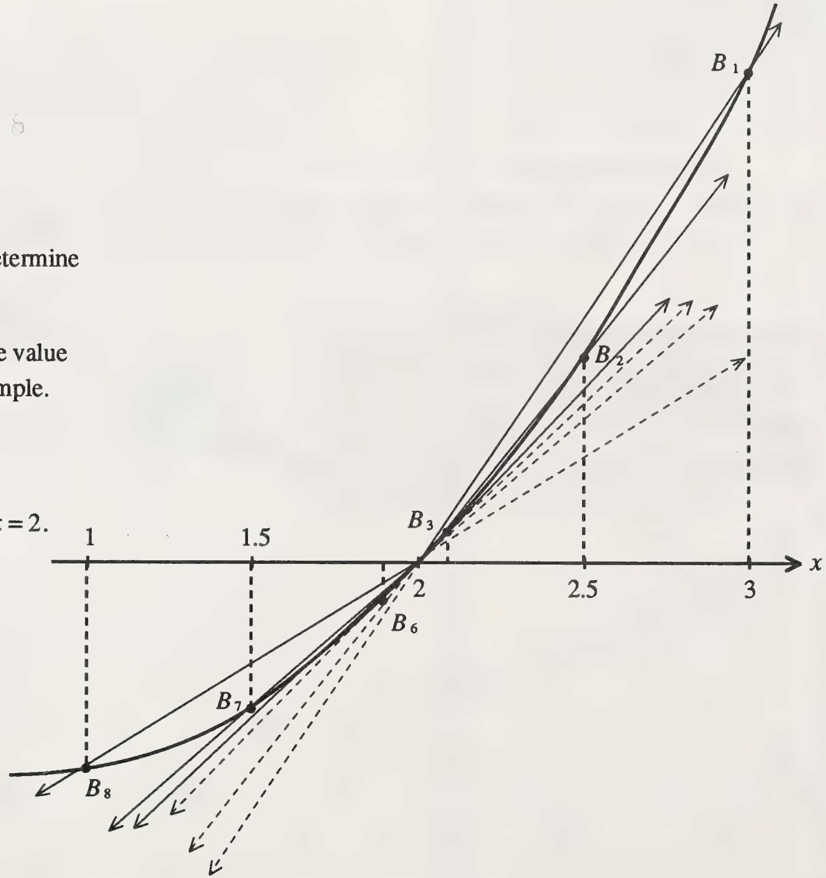
Example 4

If $y = \frac{1}{2}x^2 - 2$, find the slope of the tangent to the curve at $x = 2$.

Solution:

$$\begin{aligned} \text{When } x = 2,\ y &= \frac{1}{2}(2)^2 - 2 \\ &= 0 \end{aligned}$$

Name $(2, 0)$ point A. Choose a sequence of secants containing A and points $B_1, B_2, B_3, \ldots$ such that B_n is approaching A.



Choose values of x close to but not equal to 2. Determine the y-values for the points chosen. Then determine the slopes of the secants through these points. Summarize your findings in a table.

Coordinates of B_n			Slope of secant through $A(2,0)$ and $B_n(x,y)$
Point	x	$y=\frac{1}{2}x^2-2$	$m=\frac{y_2-y_1}{x_2-x_1}=\frac{y_1-y_2}{x_1-x_2}$
B_1	3	2.5	$\frac{0-2.5}{2-3}=2.5$
B_2	2.5	1.125	$\frac{0-1.125}{2-2.5}=2.25$
B_3	2.1	0.205	$\frac{0-0.205}{2-2.1}=2.05$
B_4	2.01	0.020 05	$\frac{0-0.020\,05}{2-2.01}=2.005$
B_5	1.99	−0.019 95	$\frac{0-(-0.019\,95)}{2-1.99}=1.995$
B_6	1.9	−0.195	$\frac{0-(-0.195)}{2-1.9}=1.95$
B_7	1.5	−0.875	$\frac{0-(-0.875)}{2-1.5}=1.75$
B_8	1	−1.5	$\frac{0-(-1.5)}{2-1}=1.5$

According to the sequence of slopes of the secants, you could conclude that the slope of the tangent at $(2,0)$ is 2. This is the **limiting value** of the sequence of slopes.

At this point you are probably questioning the usefulness of calculus.

Calculus does not seem to provide an easy way of finding the slope of a tangent, nor does it provide a solution to the problem of determining where the slope of the tangent to a curve is zero. You still need to determine how to find maximum and minimum points. More investigations have to be done before you can discover how calculus can make your calculation much easier.

Before continuing with your investigations, do at least one of the following two questions. Use the procedure shown in Example 4 to verify each of the following statements.

1. At the point where $x=1$, the slope of the tangent to $y=x^2-3$ is 2.

2. At the point where $x=1$, the slope of the tangent to $y=x^3-2x$ is 1.

For solutions to **Activity 2**, turn to the **Appendix, Topic 2**.

Activity 3

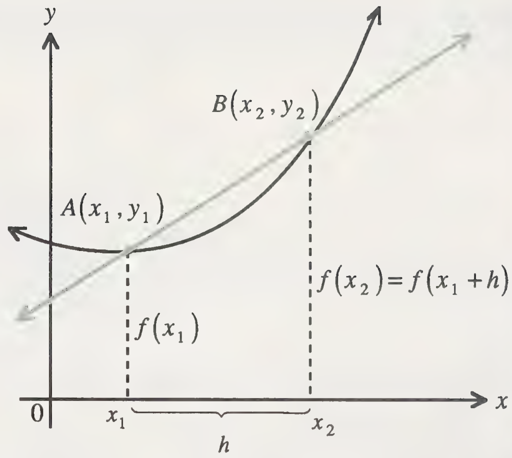
Define and determine the first derivative of a function.

In Activity 2 you did a lot of calculating. You were looking for the slope of a tangent to a curve defined by a function. You chose a sequence of points in the neighbourhood of the given point and calculated the slopes of the secant lines. The limiting value of the sequence of slopes is the slope of the tangent line. Now generalize the situation.

How do you determine the slope of the tangent to the curve $y = f(x)$ at $x = x_1$?

If $y_1 = f(x_1)$, then the tangent line touches the curve at $A(x_1, y_1)$.

If you move away from point $A(x_1, y_1)$ a horizontal distance of h units to point $B(x_2, y_2)$, then $x_2 = x_1 + h$ and $y_2 = f(x_1 + h) = f(x_2)$.



The slope of the secant line is as follows:

$$AB = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{f(x_1 + h) - f(x_1)}{(x_1 + h) - x_1}, \ h \neq 0$$
$$= \frac{f(x_1 + h) - f(x_1)}{h}$$

Use a new symbol Δx (delta x) to denote the increase in x and Δy (delta y) to denote the corresponding increase in y.

In this case,

$$\Delta x = x_2 - x_1 = h$$

$$\Delta y = y_2 - y_1 = f(x_1 + h) - f(x_1)$$

$$m_{AB} = \frac{\Delta y}{\Delta x} = \frac{f(x_1 + h) - f(x_1)}{h}$$

If point B is moving toward point A, then h is getting smaller and approaching zero. If $h = 0$, then m_{AB} is undefined; therefore, h cannot be zero. As h approaches zero, the slope of the secant approaches a value which is the slope of the tangent. Zero is the limit of h. The symbol **lim** is used to denote **limit** and the symbol $\boldsymbol{h \to 0}$ is used to denote ***h* approaches zero**.

The slope of the tangent is the limit of the slope of the secant as $h \to 0$.

$$\text{Slope of tangent} = \lim_{h \to 0} \frac{\Delta y}{\Delta x}$$

$$= \lim_{h \to 0} \frac{f(x_1 + h) - f(x_1)}{h}$$

Since x_1 represents different values of x, x will be used instead of x_1.

The value of this limit at x is also called the **derivative of *y* with respect to *x*** because it is derived from the original function. The symbol $\frac{dy}{dx}$ is used for this value.

The derivative of y with respect to x can be denoted as follows:

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{\Delta y}{\Delta x}$$

This value can also be denoted by other notations:

$D_x y$

y' (read as y prime)

$f'(x)$ (read as f prime at x)

Since $D_x y$ represents the first time you take the derivative of y, it should be called **the first derivative of *y***. (You can take the derivative of y more than once.) According to the definition of $\frac{dy}{dx}$, you can conclude that **first derivatives are slopes of tangents**.

m_{AB} = slope of secant AB

Now look at the following example to discover how you can make use of this generalization to determine the slope of the tangent to a curve.

Example 5

At the point on the curve where $x=2$, determine the slope of the tangent to the curve $y=x^2-3$.

Solution:

$$f(x)=x^2-3$$

$$\begin{aligned} f(x+h) &= (x+h)^2-3 \\ &= x^2+2hx+h^2-3 \end{aligned}$$

$$\begin{aligned} \text{Slope} = \frac{dy}{dx} &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{\left(x^2+2hx+h^2-3\right)-\left(x^2-3\right)}{h} \\ &= \lim_{h\to 0}\frac{2hx+h^2}{h} \\ &= \lim_{h\to 0}\frac{\cancel{h}(2x+h)}{\cancel{h}} \\ &= \lim_{h\to 0}(2x+h) \end{aligned}$$

You know that $h\neq 0$, but when h approaches 0, $2x+h$ tends to $2x$ so the slope of the tangent to the curve is given by $2x$.

$$\text{Slope} = \frac{dy}{dx} = 2x$$

At the point on the curve where $x=2$, the slope is as follows:

$$\begin{aligned} \text{Slope} = \frac{dy}{dx} &= 2(2) \\ &= 4 \end{aligned}$$

Where $x=2$, the slope of the tangent to the curve $y=x^2-3$ is 4.

The procedure outlined in the previous example is called **finding the derivative of a function from first principles**.

When $x=2$, $y=2^2-3=1$.

Therefore, the point is $(2,1)$.

For this question you do not need to know the y-value.

Now try a cubic function.

Example 6

- At any point on the curve, determine the slope of the tangent to the curve $y=2x^3$.

Solution:

$$\begin{aligned} f(x) &= 2x^3 \\ f(x+h) &= 2(x+h)^3 \\ &= 2\left(x^3+3hx^2+3h^2x+h^3\right) \\ &= 2x^3+6hx^2+6h^2x+2h^3 \end{aligned}$$

$$\begin{aligned} \text{Slope} = \frac{dy}{dx} &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{\cancel{2x^3}+6hx^2+6h^2x+2h^3-\cancel{2x^3}}{h} \\ &= \lim_{h\to 0}\frac{\cancel{h}\left(6x^2+6hx+2h^2\right)}{\cancel{h}} \\ &= \lim_{h\to 0}6x^2+6hx+2h^2 \\ &= 6x^2 \ \left(\text{When } h\to 0,\ 6hx\to 0 \text{ and } 2h^2\to 0.\right) \end{aligned}$$

The slope of the tangent to the curve at any point is $6x^2$.

- At any point on the curve, determine the slope of the tangent to the curve $y=-x$.

Solution:

$$\begin{aligned} f(x) &= -x \\ f(x+h) &= -(x+h) \end{aligned}$$

$$\begin{aligned} \text{Slope} = D_x y &= \lim_{h\to 0}\frac{-(x+h)-(-x)}{h} \\ &= \lim_{h\to 0}\frac{\cancel{-x}-h+\cancel{x}}{h} \\ &= -1 \end{aligned}$$

The slope of the tangent to the curve at any point is -1.

- At any point on the curve, determine the slope of the tangent to the curve $y=2x^3-x+1$.

Solution:

$$\begin{aligned} f(x) &= 2x^3-x+1 \\ f(x+h) &= 2(x+h)^3-(x+h)+1 \\ &= 2\left(x^3+3hx^2+3h^2x+h^3\right)-x-h+1 \\ &= 2x^3+6hx^2+6h^2x+2h^3-x-h+1 \end{aligned}$$

$$
\begin{aligned}
\text{Slope} &= D_x y \\
&= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{\cancel{2x^3} + 6hx^2 + 6h^2x + 2h^3 - \cancel{x} - h + \cancel{1} - \cancel{2x^3} + \cancel{x} - \cancel{1}}{h} \\
&= \lim_{h \to 0} \frac{\cancel{h}\left(6x^2 + 6hx + 2h^2 - 1\right)}{\cancel{h}} \\
&= \lim_{h \to 0} 6x^2 + 6hx + 2h^2 - 1 \\
&= 6x^2 - 1 \; \left(\text{When } h \to 0,\ 6hx \text{ and } 2h^2 \text{ tend to } 0.\right)
\end{aligned}
$$

The slope of the tangent to the curve $y = 2x^3 - x + 1$ at any point is $6x^2 - 1$.

The results in Example 6 show the following:

- The derivative of $2x^3 = 3 \times 2x^2 = 6x^2$.
- The derivative of $-x = -1 \left(-1 = -x^0\right)$.
- The derivative of $2x^3 - x + 1 = 6x^2 - 1$.

Therefore, $D_x\left(2x^3 - x + 1\right) = D_x\left(2x^3\right) + D_x(-x) + 0$.

Take a closer look at these results.

The difference is 1.

The derivative of $2x^{3} = 3 \times 2x^2 = 6x^{2}$.

same

The difference is 1.

The derivative of $-x^{1} = -1$, and $-1 = -\boxed{1}x^{0}$.

same

Do you see the pattern? By inspection, can you find the slope of the tangent to the curve $y = x^3$? If your answer is $3x^2$, you are right.

What is the derivative of $y = x^4 - 3x^2 + 5x - 7$? If your answer is as follows, you are right!

$$
\begin{aligned}
D_x y &= D_x\left(x^4\right) - D_x\left(3x^2\right) + D_x(5x) - D_x(7) \\
&= 4x^3 - 6x + 5 - 0 \\
&= 4x^3 - 6x + 5
\end{aligned}
$$

You have discovered a pattern.

The derivative of x^n is nx^{n-1}.

Notice that the exponent in the derivative has changed to one less than the original exponent. You now have x^{n-1} which is multiplied by the exponent n to give the derivative nx^{n-1}.

If you use this pattern to find the derivative of a function, your calculations are reduced. However, you have used inductive reasoning to establish this pattern and inductive reasoning is not a proof. As you learn more about limit in the next section, you will be able to apply rules for finding the derivative of a function. From first principles you can prove that $D_x ax^2 = 2ax$ and $D_x x^3 = 3x^2$, but you cannot prove that $D_x x^n = nx^{n-1}$.

You cannot assume the pattern holds true for all polynomial functions. Furthermore, there is another type of function that you have not yet studied, namely, the reciprocal function.

Consider a reciprocal function in the next example.

Example 7

At the point on the curve where $x = 5$, determine the slope of the tangent to the curve $y = \frac{5}{x}$.

Solution:

$$f(x) = \frac{5}{x}$$

$$f(x+h) = \frac{5}{(x+h)}$$

$$\begin{aligned}
\text{Slope} = \frac{dy}{dx} &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{\frac{5}{x+h} - \frac{5}{x}}{h} \\
&= \lim_{h \to 0} \frac{5(x) - 5(x+h)}{h(x)(x+h)} \\
&= \lim_{h \to 0} \frac{5x - 5x - 5h}{h(x)(x+h)} \\
&= \lim_{h \to 0} \frac{-5\cancel{h}}{\cancel{h}(x)(x+h)} \\
&= \lim_{h \to 0} \frac{-5}{x(x+h)} \\
&= \frac{-5}{x^2} \quad (\text{When } h \to 0,\ x + h \to x.) \\
&= -5x^{-2}
\end{aligned}$$

Therefore, the slope of the tangent at any point on the curve is $\frac{-5}{x^2}$. The slope of the tangent at the point on the curve where $x = 5$ is as follows:

$$\begin{aligned}
m &= \frac{dy}{dx} \\
&= -\frac{5}{(5)^2} \\
&= -\frac{1}{5}
\end{aligned}$$

Compare the function and its derivative.

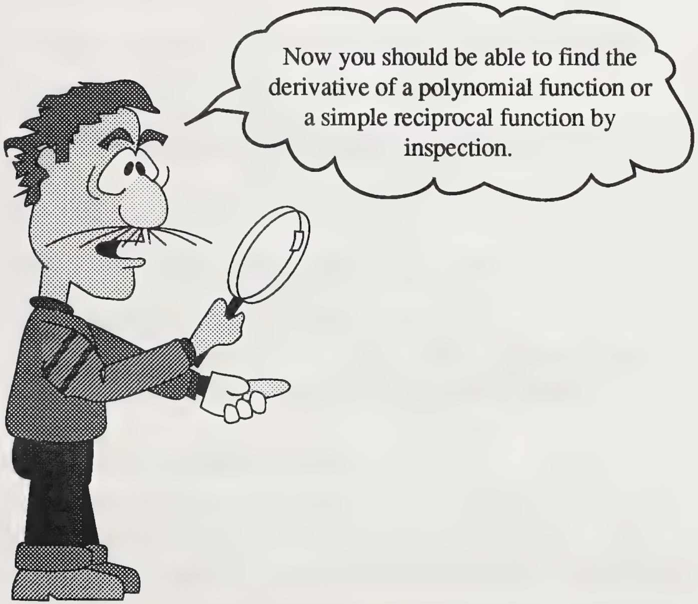
The difference is 1 since -2 is one less than -1.

If $y = \frac{5}{x} = 5x^{-1}$, then $D_x y = -1(5)x^{-2} = -5x^{-2}$.

same

same

The pattern is the same.



The derivatives you should know are shown in the following table.

Function	Derivative
$y=c$ (c is a constant.)	0
$y=x$	1
$y=bx$ (b is a constant.)	b
$y=x^2$	$2x$
$y=ax^2$	$2ax$
$y=x^n$	nx^{n-1}

Now try some questions.

Complete the even-numbered questions. Complete the others if you wish.

From first principles, verify each of the statements in questions 1 to 4.

1. If $y=2x-3$, then $\frac{dy}{dx}=2$.

2. At any point on the curve, the slope of the tangent to the curve $y=3x^2-2x+4$ is $6x-2$.

3. $y=x^3+x-1$, $\frac{dy}{dx}=3x^2+1$

4. If $y=\frac{-2}{x^2}$, then $\frac{dy}{dx}=4x^{-3}=\frac{4}{x^3}$.

Find, by inspection, the first derivative of each of the functions in questions 5 to 10.

5. x^4

6. $3x^3$

7. $2x^3+3x^2$

8. $\frac{3}{x^4}$

9. $2x+3$

10. $\frac{1}{x}+2$

11. At any point, find the slope of the tangent to the graph $y=3x^2-4x+1$.

12. At any point, find the slope of the tangent to the graph $y=x^3-2x^2$.

13. At $x=3$, find the slope of the tangent to the graph $y=x^4+x$.

14. At $x=2$, find the slope of the tangent to the graph $y=3x-2x^3$.

For solutions to **Activity 3**, turn to the **Appendix, Topic 2**.

Activity 4

Determine the equations of tangent lines to algebraic functions.

Now that you know how to find the slope of a tangent line, you should proceed to find the equation.

It is useful to recall that the derivative of the curve at a given point gives the slope of the tangent line at that point. You will use the notation $\frac{dy}{dx}$ to denote the derivative of y with respect to x.

When you have found the value of the derivative $\frac{dy}{dx}$ at a point on the graph of a relation or a function, you can find the equation of the tangent at that point.

Here are some examples related to the previous discussion.

Example 8

Find the equation of the tangent line to the curve $f(x)=5x^3-4x^2+3x-4$ at $x=-1$.

Solution:

$$f(x)=5x^3-4x^2+3x-4$$

$$\therefore \frac{dy}{dx}=15x^2-8x+3$$

$$\text{At } x=-1,\ \frac{dy}{dx}=15(-1)^2-8(-1)+3$$
$$=26$$

The slope at $x=-1$ is 26.

$Ax+By=C$ and $\frac{-A}{B}=m$

$$\frac{-A}{B}=\frac{A}{-B}$$
$$=\frac{26}{-(-1)}$$

Thus, $A=26$ and $B=-1$.

$$Ax+By=C$$
$$26x-y=C$$

When $x=-1$, $y=f(-1)=5(-1)^3-4(-1)^2+3(-1)-4$

$$=-5-4-3-4$$
$$=-16$$

The point of tangency is $(-1,-16)$.

Substitute $(-1,-16)$ for (x, y).

$$26(-1)-(-16)=C$$
$$-26+16=C$$
$$-10=C$$

Thus, $26x-y=-10$ is the equation of the tangent line.

Example 9

Find the equation of the tangent to the curve $f(x)=5x^3-4x^2+3x-4$ when $x=2$.

Solution:

$$\frac{dy}{dx}=15x^2-8x+3$$

$$\text{At } x=2,\ \frac{dy}{dx}=15(2)^2-8(2)+3$$
$$=47$$

Thus, the slope of the tangent at $x=2$ is 47.

$Ax + By = C$ and $\frac{-A}{B} = m$

$$\frac{-A}{B} = \frac{A}{-B}$$

$$= \frac{47}{-(-1)}$$

Thus, $A = 47$ and $B = -1$.

$$Ax + By = C$$

$$47x - y = C$$

When $x = 2$, $y = f(2) = 5(2)^3 - 4(2)^2 + 3(2) - 4$

$$= 40 - 16 + 6 - 4$$

$$= 26$$

The point of tangency is $(2, 26)$.

Substitute $(2, 26)$ for (x, y).

$$47(2) - 26 = C$$

$$94 - 26 = C$$

$$68 = C$$

Thus, $47x - y = 68$ is the required equation.

Example 10

Find the equation of the tangent to the curve $y = 5x^3 - 4x^2 + 3x - 4$ at $x = 0$, $y = -4$.

Solution:

The derivative of $f(x) = 5x^3 - 4x^2 + 3x - 4$ is $\frac{dy}{dx} = 15x^2 - 8x + 3$.

When $x = 0$, $\frac{dy}{dx} = 3$; thus, the slope is 3.

$Ax + By = C$ and $\frac{-A}{B} = m$

$$\frac{-A}{B} = \frac{A}{-B}$$

$$= \frac{3}{-(-1)}$$

Thus, $A = 3$ and $B = -1$.

$$Ax + By = C$$

$$3x - y = C$$

Substitute $(0, -4)$ for (x, y).

$$3(0) - (-4) = C$$

$$4 = C$$

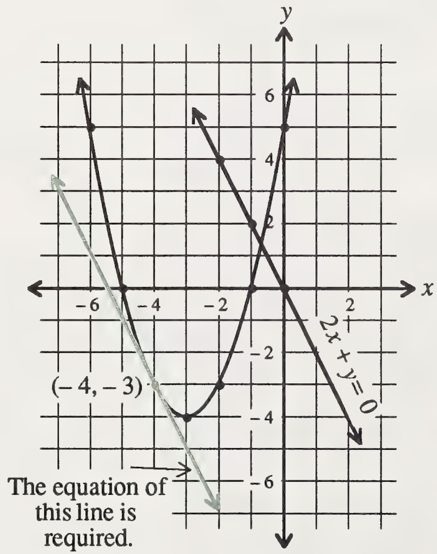
Therefore, $3x - y = 4$ is the equation of the tangent line.

Example 11

Write the equation of the line tangent to $y = x^2 + 6x + 5$ and parallel to $2x + y = 0$.

Solution:

Sketch the graph of these two equations so you can understand what is happening.



$y = x^2 + 6x + 5$

$0 = x^2 + 6x + 5$

$0 = (x+5)(x+1)$

The x-intercepts are -5 and -1.
The y-intercept is 5.

If the tangent line is parallel to $2x + y = 0$, then it is equal to $2x + y = C$ since both lines have the same slope of -2.

You need a point common to the equations $y = x^2 + 6x + 5$ and $2x + y = C$. To find that point, set the slopes equal. The slope of $y = x^2 + 6x + 5$ is $\frac{dy}{dx} = 2x + 6$. The slope of $2x + y = C$ is -2.

$$\therefore 2x + 6 = -2$$
$$x = -4$$

Now solve for y.

$$y = (16 - 24 + 5)$$
$$= -3$$

The point is $(-4, -3)$.

Now write the equation of the tangent line given $2x + y = C$ and $P(-4, -3)$.

$$2(-4) - 3 = -11$$

The equation of the tangent line is $2x + y = -11$.

Now do the following questions using your knowledge of derivatives.

1. Write the equation of the line tangent to $y = \frac{2}{x} - x^2$ at $x = -1$.
2. Write the equation of the line tangent to $y = 3x^3 - 2x$ at $x = 2$.
3. Write the equation of the line tangent to $y = x^2 + 5x + 4$ and parallel to $x + y = 1$.
4. Write the equation of the line tangent to $y = x^2 + 2x + 1$ and perpendicular to $2x - y = 1$.
5. Write the equations of the tangents to the curve defined by $y = x^2 - 8x - 8$ at its intersection points with the curve $y = -x^2 + 16$.

For solutions to **Activity 4**, turn to the **Appendix, Topic 2**.

If you require help, do the Extra Help section.

If you want more challenging explorations, do the Extensions section.

You may decide to do both.

Extra Help

A tangent may be expressed as follows:

$$\text{Tangent} = \text{slope}\ (m) = \frac{\text{rise}}{\text{run}}$$

The tangent refers to the slope of an incline.

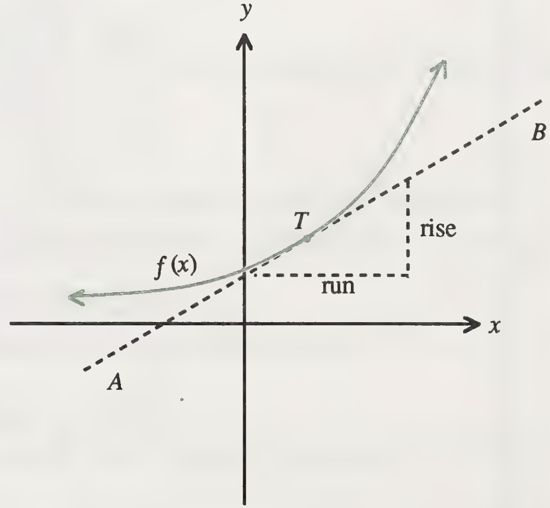
You can better understand this concept when the **rise** and the **run** are explained in terms of an angle.

The run is the distance along the horizontal. The rise is the vertical distance above the horizontal.

The angle θ is the displacement between the horizontal and the extent of the rise.

By definition $\tan\theta = \frac{\text{rise}}{\text{run}}$.

This idea is very useful when determining the change in the curvature of a function.



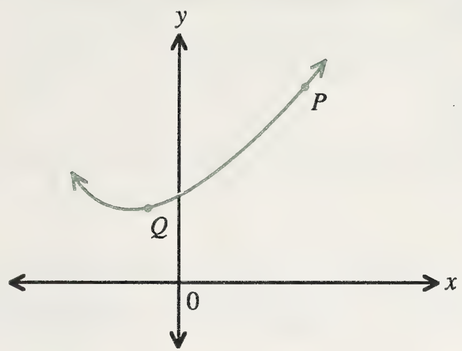
The tangent AB is touching the curve $f(x)$ at point T.
The rise over run gives the slope of the tangent line.

At the point T, the slope (m) of AB is said to be $\frac{\Delta y}{\Delta x}$.

Look at the following example.

Example 12

$Q(-1,2)$ and $P(3,5)$ are two points on a curve. Find the slope of the line segment PQ. What is the equation of PQ?



Solution:

$$\text{Slope } (m) = \frac{\Delta y}{\Delta x}$$
$$= \frac{5-2}{3-(-1)}$$
$$= \frac{3}{4}$$

The slope of PQ is $\frac{3}{4}$.

The equation of the line PQ is given by the following:

$$y-5=\frac{3}{4}(x-3)$$
$$4y-20=3x-9$$
$$3x-4y+11=0$$

Now try the following questions.

1. Find the slope and the equation of a line segment that goes through the points $A(-2,1)$ and $B(3,-4)$.

2. If $\frac{\Delta y}{\Delta x}$ of a tangent line is $-\frac{2}{3}$ and the point of tangency is $T(-3,-4)$, what is the equation of the tangent?

For solutions to **Extra Help**, turn to the **Appendix, Topic 2**.

Δy is read **delta** $\boldsymbol{y}$ which means the change in y.

Δx is read **delta** $\boldsymbol{x}$ which means the change in x.

The slope of a secant becomes the slope of a tangent when the distance between the two points of intersection approaches zero. Since slope $= \frac{\text{rise}}{\text{run}}$, the slope of a secant can be represented by $\frac{\Delta y}{\Delta x}$, where Δx is the increase in x and Δy is the corresponding increase in y. Therefore, the slope of a secant is as follows:

$$\frac{\Delta y}{\Delta x} = \frac{f(x_1 + h) - f(x_1)}{h}, \text{ where } h = \Delta x$$

The slope of a secant becomes the slope of a tangent when $\lim\limits_{h \to 0} \frac{\Delta y}{\Delta x}$, where **lim** is the symbol used to denote limit. Note that $\lim\limits_{h \to 0} \frac{\Delta y}{\Delta x}$ also can be denoted by the symbol $\frac{dy}{dx}$, $D_x y$, $f'(x)$, or y'. $D_x y$ is also called the first derivative of y with respect to x.

If a function $y = x^2 + 1$ is given, then the slope of the tangent at any point is as follows:

$$\begin{aligned}
\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \to 0} \frac{\left[(x+h)^2 + 1\right] - \left(x^2 + 1\right)}{h}, \ h \neq 0 \\
&= \lim_{h \to 0} \frac{\left(x^2 + 2hx + h^2 + 1\right) = \left(x^2 + 1\right)}{h} \\
&= \lim_{h \to 0} \frac{\cancel{x^2} + 2hx + h^2 + \cancel{1} - \cancel{x^2} - \cancel{1}}{h} \\
&= \lim_{h \to 0} \frac{2hx + h^2}{h} \\
&= \lim_{h \to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}} \\
&= \lim_{h \to 0} 2x + h \\
&= 2x
\end{aligned}$$

The preceding procedure is called finding the derivative of a function from first principles. From first principles the following statements can be verified.

- The derivative of a constant with respect to x is zero.

  $D_x c = 0$ or $\frac{dc}{dx} = 0$ or $y' = 0$

- The derivative of x with respect to x is 1.

  $D_x x = 1$ or $\frac{dx}{dx} = 1$

- The derivative of bx with respect to x is b.

  $D_x(bx) = b$ or $\frac{d(bx)}{dx} = b$

- The derivative of ax^2 with respect to x is $2ax$.

  $D_x ax^2 = 2ax$

These statements also imply that the derivative of x^n with respect to x is nx^{n-1}.

Try the following questions.

3. From first principles verify that $D_x y = 3$ for $y = 3x - 5$.

4. By inspection state the first derivative of each of the following:

   a. $y = x + 8$

   b. $y = 3x - 5$

   c. $y = x^3 - x^2 + 1$

   d. $y = \frac{1}{x^2}$

   e. $y = 3x^4 - \frac{1}{x}$

5. Determine the equation of the line tangent to $y = x^2 - 3x + 6$ and parallel to $x - y = 2$.

6. Determine the equation of the line tangent to $y = x^2 - 5x + 2$ and perpendicular to $3x + y = 2$.

For solutions to **Extra Help**, turn to the **Appendix**, **Topic 2**.

Extensions

The point on a curve $f(x)$ may be expressed definitively as the point $P(2,3)$. However, when you talk about a special point in a general way, you write $P(x_1, y_1)$. This provides a way to derive equations and formulas.

Example 13

Find the equation of the line that passes through the points $P(x_1, y_1)$ and $Q(x_2, y_2)$.

Solution:

You first find the slope *m*.

$$m = \frac{y_2 - y_1}{x_2 - x_1} \qquad \text{(This is the slope in general form.)}$$

For the equation $y - y_1 = m(x - x_1)$,

substitute $m = \dfrac{y_2 - y_1}{x_2 - x_1}$; then, $y - y_1 = \dfrac{y_2 - y_1}{x_2 - x_1}(x - x_1)$.

This is the general form for the equation of a line given two points.

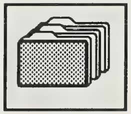
Try the following questions.

1. Find the slope of a tangent that touches the curve $f(x) = 3x^2$ at $x = 3x_1$. If the tangent also passes through $P(3, 27)$, what is the equation of the tangent?

2. The tangent to the curve $y(1 + x^2) = 2$ at the point $P(2, \frac{2}{5})$ meets the curve again at Q. If the slope of the tangent at P is $-\frac{8}{25}$, find the coordinates of Q.

For solutions to **Extensions**, turn to the **Appendix, Topic 2**.

Unit Summary



What You Have Learned

In this unit you learned the following:

- The limits of infinite sequences and algebraic functions can be determined.
- Limit theorems can be used in calculations.
- A secant is a line that intersects a curve at two distinct points.
- A tangent is a line which is at the limiting position of a sequence of secants as the distance between the points approaches zero.
- The slope of a curve at a point equals the slope of the tangent at that point and this equals the first derivative of the function.

Unit Summary

- The derivative of a function can be obtained from first principles where

  $$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x_1 + h) - f(x_1)}{h}.$$

- The slopes and equations of secants and tangents can be calculated.

You are now ready to complete the **Unit Assignment**.

Appendix

Solutions

Review

Topic 1 Limits and Derivatives

Topic 2 Secant and Tangent Lines

Review

1. a. $\frac{6x^3y^5}{8x^3y^6} = \frac{3}{4y}$

   b. $\frac{3x^{-1}y^{-2}}{-6xy^{-3}} = \frac{3y^3}{-6x^2y^2}$

   $= \frac{y}{-2x^2}$

   OR

   $\frac{3x^{-1}y^{-2}}{-6xy^{-3}} = \frac{1x^{-2}y}{-2}$

   $= \frac{y}{-2x^2}$

2. a. $x^{\frac{3}{4}}$ b. $x^{\frac{1}{2}}$

   c. $2x^{-\frac{1}{3}}$

3. a. $x^2 - 6x + 9 + 4x^2 - 8x - x^2 + 12x - 36$

   $= 4x^2 - 2x - 27$

   b. $\left(x^2 + 2xh + h^2 - x - h\right) - x^2 + x$

   $= x^2 + 2xh + h^2 - x - h - x^2 + x$

   $= 2xh + h^2 - h$

4. a. $(2x-1)(2x-1)$ or $(2x-1)^2$ b. $2h\left(2x^3h - 1\right)$

   c. $(2x+3)(2x-3)$ d. $x^{-5}\left(x^3+1\right)$

   e. $(x+1)^{-3}(x+2)$ f. $(x+2)^{-\frac{1}{2}}(x+3)$

5. a. $\frac{x(x-6)}{(x+6)(x-6)} = \frac{x}{x+6}$ (Factor first.)

   b. $\frac{2(x-3)(2x-3)}{(x-3)^2} = \frac{2(2x-3)}{x-3}$

   c. $\frac{(x+2)^{-3}\left[(x+2)^1(x-6)+3x\right]}{(x+2)^4}$

   Since $(x+2)^{-3}$ is smaller than $(x+2)^{-2}$, it is the common factor.

   $(x+2)^{-3}(x+2)^1 = (x+2)^{-2}$

   $= \frac{(x+2)^{-3}\left(x^2-4x-12+3x\right)}{(x+2)^4}$

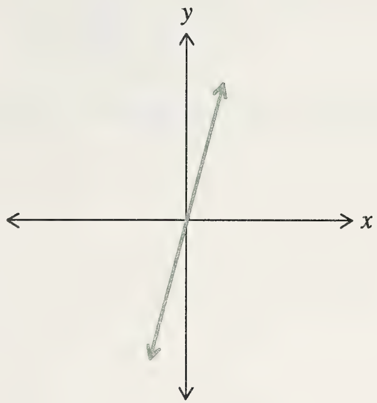
   $= \frac{x^2-x-12}{(x+2)^7}$

d. $\dfrac{3x(x+3)}{(x+1)(x+3)}+\dfrac{(x+2)(x+1)}{(x+3)(x+1)}$ (Find the L.C.D.)

$=\dfrac{3x^2+9x+x^2+3x+2}{(x+1)(x+3)}$ (Simplify the numerator.)

$=\dfrac{4x^2+12x+2}{(x+1)(x+3)}$

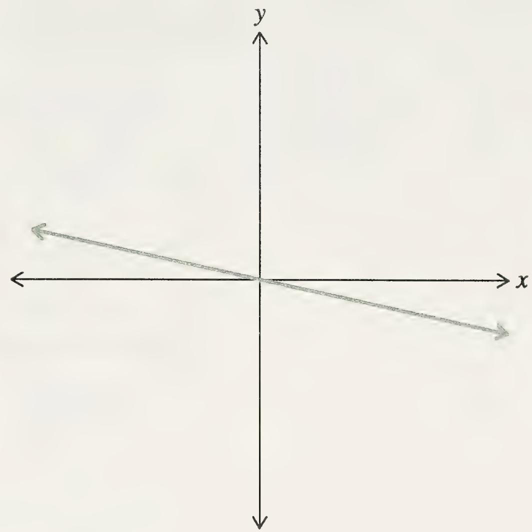
6. a.



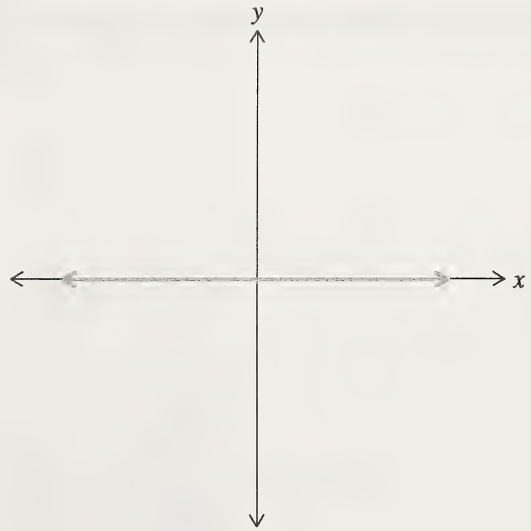
There are some basic rules to remember about slope:

- A positive (+) slope rises to the right.
- A negative (–) slope falls to the right.
- In absolute values, the bigger the number, the steeper the line will be; the smaller the number, the flatter the line will be.
- Slopes of ± 1 are at 45° to the horizontal.
- A slope of 0 (zero) is a horizontal line.
- An infinite (undefined) slope is a vertical line.

b.



c.



7. Remember that slope (m) is $\frac{\text{rise}}{\text{run}}$. The formula is $m = \dfrac{y_2 - y_1}{x_2 - x_1}$.

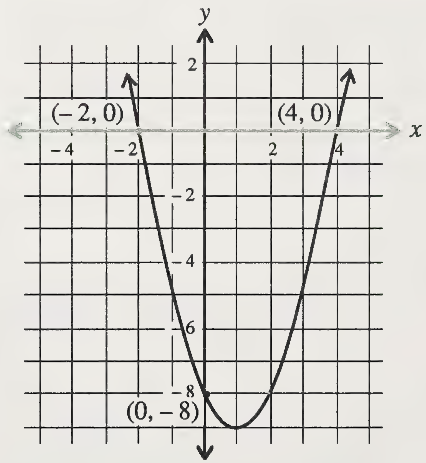
a. $$\begin{aligned} m &= \frac{0-(-3)}{3-2} \\ &= \frac{3}{1} \\ &= 3 \end{aligned}$$

b. $$\begin{aligned} m &= \frac{(x+h)^2 - (x^2)}{(x+h)-x} \\ &= \frac{x^2 + 2xh + h^2 - x^2}{h} \\ &= \frac{2xh + h^2}{h} \\ &= \frac{h(2x+h)}{h} \\ &= 2x + h \end{aligned}$$

8. a. $P(x) = (x-4)(x+2)$

zeros: 4, −2

y-intercept $= -8$

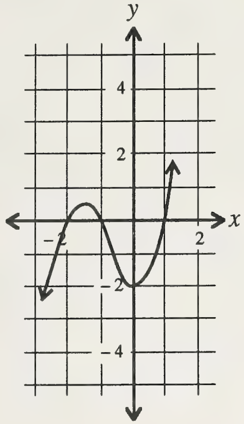


x	y
−2	0
4	0
0	−8

b. $P(x)=(x+1)(x-1)(x+2)$

zeros: $-1,\ 1,\ -2$

y-intercept $=-2$



x	y
-2	0
-1	0
0	-2
1	0

9. a. $P(2)=(2)^2-(2)+7$
$=9$

b. $P(-3)=(-3)^2-(-3)+7$
$=19$

c. $P(x+h)=(x+h)^2-(x+h)+7$
$=x^2+2xh+h^2-x-h+7$

10. $\dfrac{y-3}{x-2}=\dfrac{3-(-8)}{2-5}$ $\left(\text{since } \dfrac{y-y_1}{x-x_1}=\dfrac{y_1-y_2}{x_1-x_2}\right)$

$\dfrac{y-3}{x-2}=\dfrac{11}{-3}$

$-3y+9=11x-22$

$11x+3y-31=0$

Alternate Method:

$Ax+By=C,\ \dfrac{-A}{B}=m$

$\dfrac{y-3}{x-2}=\dfrac{11}{-3}=m$

$\dfrac{-A}{B}=\dfrac{A}{-B}$

$=\dfrac{11}{-3}$

Thus, $A=11$ and $B=3$.

$Ax+By=C$

$11x+3y=C$

Substitute $(2,3)$ for (x,y).

$11(2)+3(3)=C$

$31=C$

$11x+3y=31$

11. $m=\frac{2}{5},\ P(3,-1)$

$$\frac{y-(-1)}{x-3}=\frac{2}{5}$$
$$\frac{y+1}{x-3}=\frac{2}{5}$$
$$5y+5=2x-6$$
$$2x-5y-11=0$$

Alternate Method:

$$Ax+By=C,\ \frac{-A}{B}=m$$

$$\frac{y+1}{x-3}=\frac{2}{5}=m$$

$$\frac{-A}{B}=\frac{A}{-B}$$
$$=\frac{2}{-(-5)}$$

Thus, $A=2$ and $B=-5$.

$$Ax+By=C$$
$$2x-5y=C$$

Substitute $(3,-1)$ for (x,y).

$$2(3)-5(-1)=C$$
$$11=C$$
$$2x-5y=11$$

12. $m=\frac{1}{3},\ b=5$

$$y=mx+b$$
$$y=\frac{1}{3}x+5$$

13. a. $y=-2$

b. $x=3$

14. $3x-y-8=0$

$$y=3x-8$$
$$m=3,\ b=-8$$

a. $m=-\frac{1}{3}$

b. $m=3$

15. A line parallel to $2x+y=5$ must have the same slope. The slope of $2x+y=5$ is -2.

$$\begin{aligned} m &= \frac{-A}{B} = \frac{A}{-B} \\ &= -2 \\ &= \frac{2}{-1} \end{aligned}$$

Thus, $A=2$ and $B=1$.

$$Ax+By=C$$

$$2x+y=C \quad \text{(Substitute } (1,-2) \text{ for } (x, y).)$$

$$2(1)+(-2)=C$$

Thus, $C=0$ and the equation of the line is $2x+y=0$.

16. A line perpendicular to $3x-2y=2$ must have the negative reciprocal of the slope of $3x-2y=2$. The slope of $3x-2y=2$ is $\frac{3}{2}$. The required line has a slope equal to $-\frac{1}{\frac{3}{2}}=\frac{2}{-3}$.

$$\begin{aligned} m &= \frac{-A}{B} = \frac{A}{-B} \\ &= \frac{2}{-3} \end{aligned}$$

Thus, $A=2$ and $B=3$.

$$Ax+By=C$$

$$2x+3y=C \quad \text{(Substitute } (2,-1) \text{ for } (x, y).)$$

$$2(2)+3(-1)=C$$

Thus, $C=1$ and the equation of the required line is $2x+3y=1$.

Exploring Topic 1

Activity 1

Determine the limit of an infinite sequence.

1. a. $2, 7, 12, 17, 22$

   b. $1, -3, -7, -11, -15$

   c. $-1, 5, 15, 29, 47$

   d. $5, 14, 29, 50, 77$

2. a. $5, 4, 3\frac{2}{3}, 3\frac{1}{2}, 3\frac{2}{5}$

   b. $4, 4\frac{1}{2}, 4\frac{2}{3}, 4\frac{3}{4}, 4\frac{4}{5}$

   c. $4, 10, 28, 82, 244$

   d. $-1, -3, -7, -15, -31$

3. a. When $n=3$,

$$t_3 = -3^3$$
$$= -27$$

b. When $n=8$,

$$t_8 = (-2)^8$$
$$= 256$$

c. When $n=10$,

$$t_{10} = \frac{1}{3^{10}} - 1$$

or $$= \frac{1}{59\,049} - 1$$
$$= \frac{-59\,048}{59\,049}$$

d. When $n=9$,

$$t_9 = 3(9)^2 - 8$$
$$= 235$$

4. a. $$\lim_{n\to\infty}\left[\frac{2n-3}{n}\right] = \lim_{n\to\infty}\left[\frac{2-\frac{3}{n}}{1}\right]$$
$$= 2$$

b. $$\lim_{n\to\infty}\left[\frac{5+3n}{n}\right] = \lim_{n\to\infty}\left[\frac{5}{n}+3\right]$$
$$= 3$$

c. $$\lim_{n\to\infty}\left[\frac{3n^3-2n+1}{4n^3-2n^2+n-7}\right]$$
$$= \lim_{n\to\infty}\left[\frac{3-\frac{2}{n^2}+\frac{1}{n^3}}{4-\frac{2}{n}+\frac{1}{n^2}-\frac{7}{n^3}}\right] \quad \left(\begin{array}{l}\text{Divide both numerator and}\\ \text{denominator by } n^3.\end{array}\right)$$
$$= \frac{3}{4}$$

d. $$\lim_{n\to\infty}\left[\frac{n^3-8n^2+3n+1}{n^3-2n+8}\right]$$
$$= \lim_{n\to\infty}\left[\frac{1-\frac{8}{n}+\frac{3}{n^2}+\frac{1}{n^3}}{1-\frac{2}{n^2}+\frac{8}{n^3}}\right]$$
$$= 1$$

5. a. $\lim\limits_{n\to\infty}\left[2\left(3^n-1\right)\right]$ is a divergent sequence.

There is no limit.

b. $\lim\limits_{n\to\infty}\left[5-\left(3+2^n\right)\right]$ is a divergent sequence.

There is no limit.

c. $\lim_{n\to\infty}\left[3^{-n}+5\right]=\lim_{n\to\infty}\left[\frac{1}{3^n}+5\right]$

$=5$

d. $\lim_{n\to\infty}\left[\frac{\frac{2}{3}}{2-\left(\frac{1}{2}\right)^n}\right]=\left[\frac{\frac{2}{3}}{2-0}\right]$

$=\frac{2}{3}\div 2$

$=\frac{2}{3}\times\frac{1}{2}$

$=\frac{1}{3}$

Activity 2

Determine the limit of an algebraic function.

1. $\lim_{x\to-3}\left[\frac{x^2-9}{x+3}\right]=\lim_{x\to-3}\frac{\cancel{(x+3)}(x-3)}{\cancel{(x+3)}}$

$=-3-3$

$=-6$

2. $\lim_{x\to1}\left[\frac{x^2+7x-8}{(x-1)}\right]=\lim_{x\to1}\frac{(x+8)\cancel{(x-1)}}{\cancel{(x-1)}}$

$=1+8$

$=9$

3. $\lim_{x\to3}\left[\frac{x^2+2x-15}{x-3}\right]=\lim_{x\to3}\frac{\cancel{(x-3)}(x+5)}{\cancel{(x-3)}}$

$=3+5$

$=8$

4. $\lim_{x\to2}\left[\frac{5x-10}{x-2}\right]=\lim_{x\to2}\frac{5\cancel{(x-2)}}{\cancel{(x-2)}}$

$=5$

5. $\lim_{x\to1}\left[\frac{6x-2}{3x-1}\right]=\lim_{x\to1}\frac{2\cancel{(3x-1)}}{\cancel{(3x-1)}}$

$=2$

6. $\lim_{x\to1}\left[\frac{x-1}{x^2-1}\right]=\lim_{x\to1}\frac{\cancel{(x-1)}}{\cancel{(x-1)}(x+1)}$

$=\frac{1}{1+1}$

$=\frac{1}{2}$

7. $\lim_{x\to 2}\left[\frac{x+2}{x-3}\right]=\frac{2+2}{2-3}$

$=-4$

8. $\lim_{x\to 5}\left[\frac{x^2-x-2}{x-5}\right]=$ undefined

Therefore, there is no limit.

9. $\lim_{x\to 9}\left[\frac{x^2-1}{x-9}\right]=$ undefined (Division by zero is undefined and $x-9=0$.)

10. $\lim_{x\to 0}\left[\frac{3-\frac{1}{x}}{\frac{2}{x}-5}\right]=\lim_{x\to 0}\left[\frac{\frac{3x-1}{x}}{\frac{2-5x}{x}}\right]$

$=\lim_{x\to 0}\frac{3x-1}{2-5x}$

$=\frac{-1}{2}$

11. $\lim_{x\to 3}\left[\frac{x^3-27}{x-3}\right]=\lim_{x\to 3}\frac{\cancel{(x-3)}\left(x^2+3x+9\right)}{\cancel{x-3}}$

$=\lim_{x\to 3}x^2+3x+9$

$=27$

12. $L=\lim_{x\to 4}\frac{x^2-16}{x-4}$

$=\lim_{x\to 4}\frac{\cancel{(x-4)}(x+4)}{\cancel{(x-4)}}$

$=8$

$|f(x)-L|<10^{-6}$

If $x\neq 4, \left|\frac{x^2-16}{x-4}-8\right|<10^{-6}$

$|x+4-8|<10^{-6}$

$|x-4|<10^{-6}$

$4-10^{-6}<x<4+10^{-6}$

13. $L=\lim_{x\to 5}\frac{x^2-25}{x+5}$

$=\lim_{x\to 5}\frac{\cancel{(x+5)}(x-5)}{\cancel{(x+5)}}$

$=0$

$|f(x)-L|<10^{-8}$

If $x\neq -5$, then $\left|\frac{x^2-25}{x+5}-0\right|<10^{-8}$

$|x-5|<10^{-8}$

$5-10^{-8}<x<5+10^{-8}$

Activity 3

Use the limit theorems.

1.

LS	RS
$\lim\limits_{x\to 5}\left[\dfrac{x^2-25}{x+5}\right]\left[\dfrac{x^2-25}{x-5}\right]$	$\lim\limits_{x\to 5}\dfrac{x^2-25}{x+5}\bullet\lim\limits_{x\to 5}\dfrac{x^2-25}{x-5}$
$\lim\limits_{x\to 5}\left[\dfrac{\cancel{(x+5)}\cancel{(x-5)}(x+5)(x-5)}{\cancel{(x+5)}\cancel{(x-5)}}\right]$	$\lim\limits_{x\to 5}\dfrac{\cancel{(x+5)}(x-5)}{\cancel{(x+5)}}\bullet\lim\limits_{x\to 5}\dfrac{(x+5)\cancel{(x-5)}}{\cancel{(x-5)}}$
$\lim\limits_{x\to 5}(x+5)(x-5)$	$0(10)$
0	0

LS = RS

2.

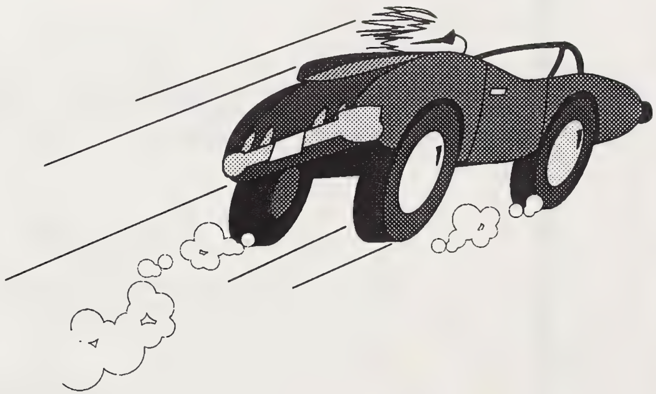
LS	RS
$\lim\limits_{x\to3}\left[\dfrac{\frac{x^2-9}{x+3}}{\frac{x^2-9}{x-3}}\right]$	$\dfrac{\lim\limits_{x\to3}\left[\frac{x^2-9}{x+3}\right]}{\lim\limits_{x\to3}\left[\frac{x^2-9}{x-3}\right]}$
$\lim\limits_{x\to3}\left[\dfrac{\frac{\cancel{(x+3)}(x-3)}{\cancel{x+3}}}{\frac{(x+3)\cancel{(x-3)}}{\cancel{(x-3)}}}\right]$	$\dfrac{\lim\limits_{x\to3}\frac{\cancel{(x+3)}(x-3)}{\cancel{(x+3)}}}{\lim\limits_{x\to3}\frac{(x+3)\cancel{(x-3)}}{\cancel{(x-3)}}}$
$\lim\limits_{x\to3}\dfrac{x-3}{x+3}$	$\dfrac{\lim\limits_{x\to3}(x-3)}{\lim\limits_{x\to3}(x+3)}$
$\dfrac{0}{6}$	$\dfrac{0}{6}$
0	0

LS = RS

3.

LS	RS
$\lim\limits_{x\to2}(2)\left(\dfrac{x^2-4}{x-2}\right)$	$2\lim\limits_{x\to2}\dfrac{x^2-4}{x-2}$
$\lim\limits_{x\to2}(2)\dfrac{(x+2)\cancel{(x-2)}}{\cancel{(x-2)}}$	$2\lim\limits_{x\to2}\dfrac{(x+2)\cancel{(x-2)}}{\cancel{(x-2)}}$
$\lim\limits_{x\to2}2(x+2)$	$2(2+2)$
$2(2+2)$	8
8	8

LS = RS



4.

LS	RS
$\lim\limits_{x\to1}\left[\dfrac{x^2-4x+3}{x-1}-\dfrac{x^2+x-2}{x-1}\right]$	$\lim\limits_{x\to1}\left[\dfrac{x^2-4x+3}{x-1}\right]-\lim\limits_{x\to1}\left[\dfrac{x^2+x-2}{x-1}\right]$
$\lim\limits_{x\to1}\left[\dfrac{\cancel{(x-1)}(x-3)}{\cancel{(x-1)}}-\dfrac{(x+2)\cancel{(x-1)}}{\cancel{(x-1)}}\right]$	$\lim\limits_{x\to1}\dfrac{\cancel{(x-1)}(x-3)}{\cancel{(x-1)}}-\lim\limits_{x\to1}\dfrac{(x+2)\cancel{(x-1)}}{\cancel{(x-1)}}$
$\lim\limits_{x\to1}[(x-3)-(x+2)]$	$\lim\limits_{x\to1}(x-3)-\lim\limits_{x\to1}(x+2)$
$\lim\limits_{x\to1}(-5)$	$(1-3)-(1+2)$
-5	-5

LS = RS

Extra Help

1. a. $\lim\limits_{n\to\infty}\left[\dfrac{n-3}{n+2}\right]=\lim\limits_{n\to\infty}\dfrac{1-\dfrac{3}{n}}{1+\dfrac{2}{n}}$ $\qquad\left(\dfrac{3}{\infty}=0,\ \dfrac{2}{\infty}=0\right)$

$$=\frac{1}{1}$$

$$=1$$

b. $\lim\limits_{n\to\infty}\left[5-\dfrac{1}{n^2}\right]=5-0$

$$=5$$

c. $$\lim_{x\to 8}\frac{\left(x^2-64\right)}{x-8}=\lim_{x\to 8}\frac{(x+8)\cancel{(x-8)}}{\cancel{(x-8)}}$$
$$=\lim_{x\to 8}(x+8)$$
$$=8+8$$
$$=16$$

d. $$\lim_{x\to 3}\left[\left(\frac{x}{3}\right)\left(\frac{x^2-9}{x-3}\right)\right]=\lim_{x\to 3}\left(\frac{x}{3}\right)\frac{(x+3)\cancel{(x-3)}}{\cancel{(x-3)}}$$
$$=\left(\frac{3}{3}\right)(3+3)$$
$$=6$$

2. $f(x)=x+5$
$$L=\lim_{x\to 3}(x+5)$$
$$=3+5$$
$$=8$$

$$|f(x)-L|<10^{-2}$$
$$|(x+5)-8|<10^{-2}$$
$$|x-3|<10^{-2}$$
$$3-10^{-2}<x<3+10^{-2}$$

$1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\ . \ . \ .$ is an infinite geometric series.

$a=1,\ r=\frac{1}{3}$

$$\therefore S_n=\frac{1\left[1-\left(\frac{1}{3}\right)^n\right]}{1-\frac{1}{3}}$$
$$=\frac{1-\frac{1}{3^n}}{1-\frac{1}{3}}$$
$$=\frac{1-\frac{1}{3^n}}{\frac{2}{3}}$$

$$L=\lim_{n\to\infty}S_n=\frac{a}{1-r}$$
$$=\frac{1}{1-\frac{1}{3}}$$
$$=\frac{3}{2}$$

$$|S_n - L| = \left| \frac{1 - \frac{1}{3^n}}{\frac{2}{3}} - \frac{3}{2} \right|$$

$$= \left| \cancel{\frac{3}{2}} - \frac{1}{2 \times 3^{n-1}} - \cancel{\frac{3}{2}} \right|$$

$$= \left| \frac{-1}{2 \times 3^{n-1}} \right|$$

Since $|S_n - L| < 3^{-9}$, then

$$\left| \frac{-1}{(2)(3)^{n-1}} \right| < 3^{-9}$$

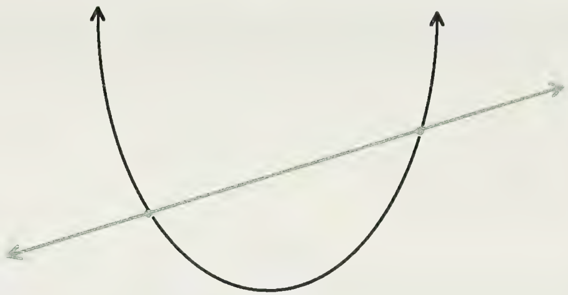
$\therefore n = 10$

Exploring Topic 2

Activity 1

Identify secant lines by finding their slopes and writing their equations.

1. a.



b.

2. a. You cannot draw a secant line because by definition, the secant line must intersect the curve in at least two places.

b. AB is a tangent. It only touches the curve, and a secant must intersect the curve in two or more places.
PQ intersects the curve, but only at one point. It must intersect the curve at least twice before it can be considered a secant.

3. $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{7-(-2)}{2-1}$

$$= \frac{9}{1}$$
$$= 9$$

The slope of the secant AB is 9.

4. Since you need to use the formula $m = \frac{y_2 - y_1}{x_2 - x_1}$, you need to find y_1 and y_2.

When $x_1 = 1$, $y_1 = 5(1)^3 - 3(1)^2 - 4(1) + 7$
$$= 5$$

When $x_2 = -1$, $y_2 = 5(-1)^3 - 3(-1)^2 - 4(-1) + 7$
$$= 5(-1) - 3(1) + 4 + 7$$
$$= -5 - 3 + 11$$
$$= 3$$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{3-5}{-1-1}$$
$$= \frac{-2}{-2}$$
$$= 1$$

The slope of the secant is 1.

5. Since $y = x^2 - 3$ and $y_1 = 1$ and $y_2 = 6$, you can find x_1 and x_2.

When $y_1 = 1$, $1 = x_1^2 - 3$
$$x_1^2 = 1 + 3$$
$$x_1^2 = 4$$
$$x_1 = \pm 2$$

When $y_2 = 6$, $6 = x_2^2 - 3$
$$x_2^2 = 6 + 3$$
$$x_2^2 = 9$$
$$x_2 = \pm 3$$

In order to get the equation, use the two-point formula $\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$ for the points.

$(2, 1)$ and $(3, 6)$ Equation I or
$(-2, 1)$ and $(-3, 6)$ Equation II

Equation I
$$\frac{y-1}{x-2} = \frac{6-1}{3-2}$$
$$\frac{y-1}{x-2} = \frac{5}{1}$$
$$y - 1 = 5x - 10$$
$$5x - y - 9 = 0$$

Equation II

$$\frac{y-1}{x-(-2)}=\frac{6-1}{-3-(-2)}$$
$$\frac{y-1}{x+2}=\frac{5}{-1}$$
$$-y+1=5x+10$$
$$5x+y+9=0$$

6. You will use $m=\dfrac{y_2-y_1}{x_2-x_1}$ to make a table showing the slope of each secant.

a.

x	y	Slope of secant through (x, y) and $(2, 4)$
1.9	$(1.9)^2$	$\frac{4-3.61}{2-1.9}=\frac{0.39}{0.1}=3.9$
1.99	$(1.99)^2$	$\frac{4-3.9601}{2-1.99}=\frac{0.0399}{0.01}=3.99$
1.999	$(1.999)^2$	$\frac{4-3.996\,001}{2-1.999}=\frac{0.003\,999}{0.001}=3.999$
.	.	
.	.	
.	.	

b. Since the slopes of the secants are 3.9, 3.99, 3.999, . . . as they approach A, the likely slope of the line at $A(2,4)$ seems to be 4.

c. The slope of the line as a multiple of a at (a, b) is $2a$.

Activity 2

Define and identify tangent lines.

1. When $x=1$, $y=-2$.
   Call $(1,-2)$ point A, and call (x, y) point B_n.
   Choose values of x greater or less than 1.

Coordinates of B_n			Slope of secant through $(1,-2)$ and $B_n(x, y)$
Point	x	$y=x^2-3$	$m=\frac{y_2-y_1}{x_2-x_1}$
B_1	1.5	−0.75	$\frac{-2-(-0.75)}{1-1.5}=2.5$
B_2	1.1	−1.79	$\frac{-2-(-1.79)}{1-1.1}=2.1$
B_3	1.01	−1.979 9	$\frac{-2-(-1.9799)}{1-1.01}=2.01$
B_4	1.001	−1.997 999	$\frac{-2-(-1.997\,999)}{1-1.001}=2.001$
B_5	0.999	−2.001 999	$\frac{-2-(-2.001\,999)}{1-0.999}=1.999$
B_6	0.99	−2.019 9	$\frac{-2-(-2.0199)}{1-0.99}=1.99$
B_7	0.9	−2.19	$\frac{-2-(-2.19)}{1-0.9}=1.9$
B_8	0.5	−2.75	$\frac{-2-(-2.75)}{1-0.5}=1.5$

According to the sequence of slopes of the secants, you could conclude that the slope of the tangent at $(1,-2)$ is 2.

2. When $x=1$, $y=x^3-2x=-1$.

   Call $(1,-1)$ point A and (x, y) point B_n.

   Choose values of x greater or less than 1.

Coordinates of B_n			Slope of secant through $(1,-1)$ and $B_n(x, y)$
Point	x	$y=x^3-2x$	$m=\frac{y_2-y_1}{x_2-x_1}$
B_1	0.5	−0.875	$\frac{-1-(-0.875)}{1-0.5}=-0.25$
B_2	0.9	−1.071	$\frac{-1-(-1.071)}{1-0.9}=0.71$
B_3	0.99	−1.009 701	$\frac{-1-(-1.009\,701)}{1-0.99}=0.9701$
B_4	0.999	−1.000 997 001	$\frac{-1-(-1.000\,997\,001)}{1-0.999}=0.997\,001$
B_5	1.001	−0.998 996 999	$\frac{-1-(-0.998\,996\,999)}{1-1.001}=1.003\,001$
B_6	1.01	−0.989 699	$\frac{-1-(-0.989\,699)}{1-1.01}=1.0301$
B_7	1.1	−0.869	$\frac{-1-(-0.869)}{1-1.1}=1.31$
B_8	1.5	0.375	$\frac{-1-0.375}{1-1.5}=2.75$

According to the sequence of slope of the secants, you could conclude that the slope of the tangent at $(1,-1)$ is 1.

Activity 3

Define and determine the first derivative of a function.

1. If $y = 2x - 3$, then $f(x) = 2x - 3$.

$$f(x+h) = 2(x+h) - 3$$

$$\begin{aligned}\frac{dy}{dx} &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\ &= \lim_{h\to 0}\frac{[2(x+h)-3]-(2x-3)}{h}\\ &= \lim_{h\to 0}\frac{\cancel{2x}+2h-\cancel{3}-\cancel{2x}+\cancel{3}}{h}\\ &= \lim_{h\to 0}\frac{2h}{h}\\ &= \lim_{h\to 0} 2\\ &= 2\end{aligned}$$

2. $f(x) = 3x^2 - 2x + 4$

$$\begin{aligned}f(x+h) &= 3(x+h)^2 - 2(x+h) + 4\\ &= 3x^2 + 6hx + 3h^2 - 2x - 2h + 4\end{aligned}$$

$$\begin{aligned}\frac{dy}{dx} &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\ &= \lim_{h\to 0}\frac{\left(3x^2+6hx+3h^2-2x-2h+4\right)-\left(3x^2-2x+4\right)}{h}\\ &= \lim_{h\to 0}\frac{6hx+3h^2-2h}{h}\\ &= \lim_{h\to 0}\frac{\cancel{h}(6x+3h-2)}{\cancel{h}}\\ &= 6x-2\end{aligned}$$

3. $f(x) = x^3 + x - 1$

$$\begin{aligned}f(x+h) &= (x+h)^3 + (x+h) - 1\\ &= x^3 + 3hx^2 + 3h^2x + h^3 + x + h - 1\end{aligned}$$

$$\begin{aligned}\frac{dy}{dx} &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\ &= \lim_{h\to 0}\frac{\left(x^3+3hx^2+3h^2x+h^3+x+h-1\right)-\left(x^3+x-1\right)}{h}\\ &= \lim_{h\to 0}\frac{3hx^2+3h^2x+h^3+h}{h}\\ &= \lim_{h\to 0}\frac{\cancel{h}\left(3x^2+3hx+h^2+1\right)}{\cancel{h}}\\ &= 3x^2+1\end{aligned}$$

4. $f(x) = -\frac{2}{x^2}$

$$f(x+h) = \frac{-2}{(x+h)^2} = \frac{-2}{x^2+2hx+h^2}$$

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{\frac{-2}{x^2+2hx+h^2}-\left(-\frac{2}{x^2}\right)}{h}$$

$$= \lim_{h\to 0}\frac{2\left(\frac{x^2+2hx+h^2-x^2}{(x^2)(x^2+2hx+h^2)}\right)}{h}$$

$$= \lim_{h\to 0}\frac{2\left(\frac{2hx+h^2}{(x^2)(x^2+2hx+h^2)}\right)}{h}$$

$$= \lim_{h\to 0}\frac{2\cancel{h}(2x+h)}{\cancel{h}(x^2)(x^2+2hx+h^2)}$$

$$= \frac{2(2x)}{(x^2)(x^2)}$$

$$= \frac{4x}{x^4}$$

$$= \frac{4}{x^3}$$

5. $\frac{dy}{dx} = 4x^{4-1}$

$= 4x^3$

6. $\frac{dy}{dx} = (3)(3)x^{3-1}$

$= 9x^2$

7. $\frac{dy}{dx} = 3(2)x^2 + 2(3)x^1$

$= 6x^2 + 6x$

8. $y = \frac{3}{x^4}$

$= 3x^{-4}$

$\frac{dy}{dx} = (-4)(3)x^{-4-1}$

$= -12x^{-5}$

or $= \frac{-12}{x^5}$

9. $\frac{dy}{dx} = 2$

10. $y = \frac{1}{x} + 2$

$= x^{-1} + 2$

$\frac{dy}{dx} = (-1)x^{-1-1} + 0$

$= -x^{-2}$

or $= \frac{-1}{x^2}$

11. $\frac{dy}{dx} = 2(3)x - 4$

$= 6x - 4$

Therefore, the slope of the tangent at any point is $6x - 4$.

12. $\frac{dy}{dx} = 3x^2 - 4x$

Therefore, the slope of the tangent at any point is $3x^2 - 4x$.

13. The slope is $\frac{dy}{dx} = 4x^3 + 1$.

At $x = 3$, slope $= 4(3)^3 + 1$

$= 108 + 1$

$= 109$

Therefore, the slope of the tangent at $x = 3$ is 109.

14. Slope $= \frac{dy}{dx} = 3 - 6x^2$

At $x = 2$, slope $= 3 - 6(2)^2$

$= 3 - 24$

$= -21$

Therefore, the slope of the tangent at $x = 2$ is -21.

Activity 4

Determine the equations of tangent lines to algebraic functions.

1. $P(-1, -3)$, $y' = \frac{-2}{x^2} - 2x$

At $x = -1$, $m = 0$; therefore, the equation is $y = -3$.

2. $y = 3(8) - 4 = 20$, $y' = 9x^2 - 2$, $m = 34$ when $x = 2$

Thus, $m = \frac{-A}{B} = \frac{A}{-B} = \frac{34}{-(-1)}$. Therefore,

$A = 34$ and $B = -1$. Substitute these values in $Ax + By = C$.

$34x - y = C$

Now substitute $(2, 20)$ for (x, y).

$34(2) - 20 = 48 = C$

The equation of the tangent is $34x - y = 48$.

3. Since the tangent is parallel to $x+y=1$, it has the form $x+y=C$. Both have the same slope of -1.

The slope of $y=x^2+5x+4$ is $\frac{dy}{dx}=2x+5$.

At the common point the slopes are equal.

$$2x+5=-1$$
$$x=-3$$

When $x=-3$, $y=(-3)^2+5(-3)+4$
$$=-2$$

The point is $(-3,-2)$. Substitute $(-3,-2)$ in $x+y=C$.

$$-3-2=C$$

Thus, $C=-5$.

The equation of the tangent is $x+y=-5$.

4. The slope of $y=x^2+2x+1$ is $\frac{dy}{dx}=2x+2$.

The slope of $2x-y=1$ is 2. Since the tangent is perpendicular to $2x-y=1$, the slope of the tangent is $\frac{1}{-2}$ (the negative reciprocal of 2.)

$$m=\frac{-A}{B}=\frac{A}{-B}$$
$$=\frac{1}{-2}$$

Thus, $A=1$ and $B=2$.

$$Ax+By=C$$
$$x+2y=C$$

At the common point the slopes are equal.

$$2x+2=-\frac{1}{2}$$

Thus, $x=\frac{-5}{4}$.

When $x=\frac{-5}{4}$, $y=\left(\frac{-5}{4}\right)^2+2\left(\frac{-5}{4}\right)+1$
$$=\frac{1}{16}$$

Substitute $\left(\frac{-5}{4},\frac{1}{16}\right)$ for (x, y).

The equation of the tangent line is $x+2y=\frac{-9}{8}$ or $8x+16y=-9$.

5. The first step is to find where the two equations intersect. Thus, set the equations equal. (It would be beneficial to graph these two equations.)

$$x^2-8x-8=-x^2+16$$
$$2x^2-8x-24=0$$
$$x^2-4x-12=0$$
$$(x-6)(x+2)=0$$

$$x=6,\ x=-2$$

Use $y=-x^2+16$ to substitute in $x=6$ and $x=-2$.

The points where the equations intersect when graphed are $(6,-20)$ and $(-2,12)$.

Since $y = -2x$, the value of the slope is $-2(6) = -12$ and $-2(-2) = 4$.

When the slope is -12, then $-12 = m = \frac{-A}{B} = \frac{A}{-B} = \frac{12}{-1}$.

Thus, $A = 12$ and $B = 1$.

$$Ax + By = C$$
$$12x + y = C$$

Now substitute $(6,-20)$ for (x, y).

$$12(6) + (-20) = C$$

Thus, $C = 52$. At point $(6,-20)$ the equation of the tangent is $12x + y = 52$.

When the slope is 4, then $4 = m = \frac{-A}{B} = \frac{A}{-B} = \frac{4}{-(-1)}$.

Thus, $A = 4$ and $B = -1$.

$$Ax + By = C$$
$$4x - y = C$$

Now substitute $(-2,12)$ for (x, y).

$$4(-2) - 12 = C$$

Thus, $C = -20$. At point $(-2,12)$ the equation of the tangent line is $4x - y = -20$.

Extra Help

1. $$\begin{aligned}\frac{\Delta y}{\Delta x} &= \frac{-4-1}{3-(-2)}\\ &= \frac{-5}{5}\\ &= -1\end{aligned}$$

   The slope of AB is -1.

   $$\begin{aligned}y-(-4) &= -1(x-3)\\ y+4 &= -x+3\\ x+y+1 &= 0\end{aligned}$$

   The equation of the line segment is $x + y + 1 = 0$.

2. $\frac{\Delta y}{\Delta x} = -\frac{2}{3}$ and the point of tangency is $(-3,-4)$.

   Thus, $$\begin{aligned}y-(-4) &= -\frac{2}{3}[x-(-3)]\\ y+4 &= -\frac{2}{3}(x+3)\\ 3y+12 &= -2x-6\\ 2x+3y+18 &= 0\end{aligned}$$

   The equation of the tangent is $2x + 3y + 18 = 0$ or $2x + 3y = -18$.

3. $f(x)=3x-5$

$$f(x+h)=3(x+h)-5$$
$$=3x+3h-5$$

$$D_x y=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$=\lim_{h\to 0}\frac{(3x+3h-5)-(3x-5)}{h}$$
$$=\lim_{h\to 0}\frac{3\cancel{h}}{\cancel{h}}$$
$$=3$$

4. a. $\frac{dy}{dx}=1$

b. $\frac{dy}{dx}=3$

c. $\frac{dy}{dx}=3x^2-2x$

d. $y=x^{-2}$

$$\therefore\frac{dy}{dx}=-2x^{-3}$$

e. $y=3x^4-x^{-1}$

$$\frac{dy}{dx}=(4)(3)x^{4-1}-(-1)x^{(-1)-1}$$
$$=12x^3+x^{-2}$$

5. Since the tangent is parallel to $x-y=2$, it has the form $x-y=C$. Both have the same slope of 1.

The slope of $y=x^2-3x+6$ is $\frac{dy}{dx}=2x-3$.

At the common point the slopes are equal.

$$2x-3=1$$
$$x=2$$

When $x=2$, $y=(2)^2-3(2)+6$
$$=4$$

The common point is $(2,4)$. Substitute $(2,4)$ in $x-y=C$.

$$2-4=C$$

Thus, $C=-2$.

The equation of the tangent is $x-y=-2$.

6. The slope of $y=x^2-5x+2$ is $\frac{dy}{dx}=2x-5$.

The slope of $3x+y=2$ is -3. Since the tangent is perpendicular to $3x+y=2$, the slope of the tangent is $\frac{1}{3}$ (the negative reciprocal of -3).

$$m=\frac{-A}{B}=\frac{A}{-B}$$
$$=\frac{1}{-(-3)}$$

Thus, $A=1$ and $B=-3$.

$$Ax+By=C$$
$$1x-3y=C$$

At the common point the slopes are equal.

$$2x-5=\frac{1}{3}$$

Thus, $x=\frac{8}{3}$.

$$y=\left(\frac{8}{3}\right)^2-5\left(\frac{8}{3}\right)+2$$
$$=\frac{-38}{9}$$

Substitute $\left(\frac{8}{3},\frac{-38}{9}\right)$ for (x,y).

$$\frac{8}{3}-3\left(\frac{-38}{9}\right)=C$$
$$\frac{46}{3}=C$$

The equation of the tangent line is $x-3y=\frac{46}{3}$ or $3x-9y=46$.

Extensions

1. When $x=3x_1$ in $f(x)=3x^2$, $y'=3(3x_1)^2$.

   The point of tangency is $T\left(3x_1, 27x_1^2\right)$.

   The slope (m) of the tangent TP through $(3, 27)$ is as follows:

$$m=\frac{27-27x_1^2}{3-3x_1}$$
$$=\frac{27\left(1-x_1^2\right)}{3(1-x_1)}$$
$$=\frac{9(1+x_1)(1-x_1)}{(1-x_1)}$$
$$=9(x_1+1)$$

   The equation of the tangent would be as follows:

$$y-27=9(x_1+1)(x-3)$$
$$y=9(x_1+1)(x-3)+27$$

2. Since the slope of the tangent to the curve at $P\left(2, \frac{2}{5}\right)$ is $-\frac{8}{25}$, the equation of the tangent is $y - \frac{2}{5} = -\frac{8}{25}(x-2)$ or $8x + 25y - 26 = 0$.

Since the tangent meets the curve again at Q, you can find the coordinates of Q by equating the y-value for the tangent line to the y-value of the given curve and then solving for x and y.

Write $y = \dfrac{2}{1+x^2}$ (1)

and $y = \dfrac{26-8x}{25}$ (2)

Equate (1) and (2).

$$\frac{26-8x}{25} = \frac{2}{1+x^2}$$

This simplifies as follows:

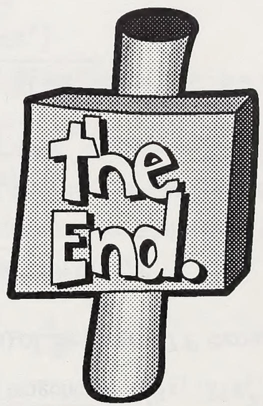
$$4x^3 - 13x^2 + 4x + 12 = 0$$

$$\text{or } (x-2)(x-2)(4x+3) = 0$$

$$\therefore x = 2 \text{ or } x = -\frac{3}{4}$$

By substituting $x = -\frac{3}{4}$ in (1), you get $y = \frac{32}{25}$.

The coordinates of Q are $\left(-\frac{3}{4}, \frac{32}{45}\right)$.



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